.

**LECTURE NOTES**
**in**
**CALCULUS I**

.

**Tadeusz STYŠ**
**MSc & Ph.D., University of Warsaw**
**Email: tadeuszstys08@gmail,com**

**Warsaw , January 2010**

# PREFACE

These lecture notes are designed for undergraduate students as a complementary reading text to a First Course in Calculus. It is assumed that the students have basic knowledge in an introductory course to Mathematics in a Science Programme.

The text book has its origin from lecture notes for courses given to undergraduate science students. The lecture notes contain short and rigorous proofs of theorems and fundamental formulas of calculus supported by example with sets of questions.

Tadeusz STYŠ

# Contents

# Chapter 1

# Integration

## 1.1 Definite Integral

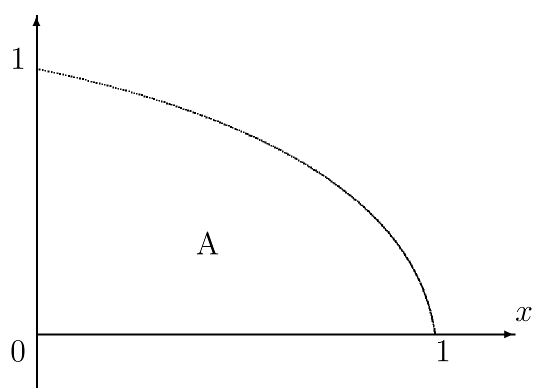
The concept of the definite integral (Riemann Integral) of a function $f(x), \ a \leq x \leq b$ is different than the notion of the indefinite integral

$$F(x) = \int f(x)dx$$

as the antiderivative $F(x)$ to the function $f(x)$.
In order to define the definite integral, we shall follow the Archimedes method (287-212 B.C.) of evaluation of the area below graph of a function $f(x), \ a \leq x \leq b$. Let us start with the example

**Example 1.1** *Find the area between x axis and the graph of the function*

$$f(x) = 1 - x^2, \qquad 0 \leq x \leq 1.$$

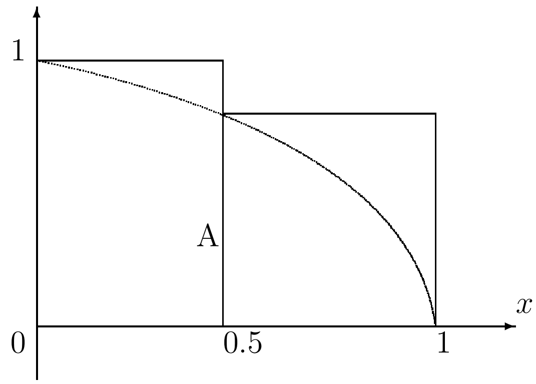



Area $I(f)$

Let us divide the interval $[0, 1]$ in two subintervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ (n=2) and evaluate the approximate value of the area $A$ by using upper sums (see Fig.

1)

$$A \approx f(0)\frac{1}{2} + f(\frac{1}{2})\frac{1}{2} + f(1)1 = 1 * \frac{1}{2} + \frac{3}{4} * \frac{1}{2} = \frac{7}{8} = 0.875$$



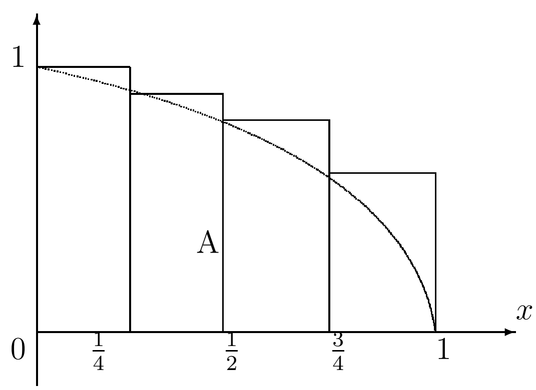

Area $I(f)$

Fig 1. $x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$

$$A \approx f(0)\frac{1}{2} + f(\frac{1}{2})\frac{1}{2} + f(1)1 = 1 * \frac{1}{2} + \frac{3}{4} * \frac{1}{2} = \frac{7}{8} = 0.875$$

Now, let us divide the interval $[0, 1]$ in four subintervals

$$[0, 1/4],\ [1/4, 1/2],\ [1/2, 3/4],\ [3/4, 1], \qquad (n = 4).$$



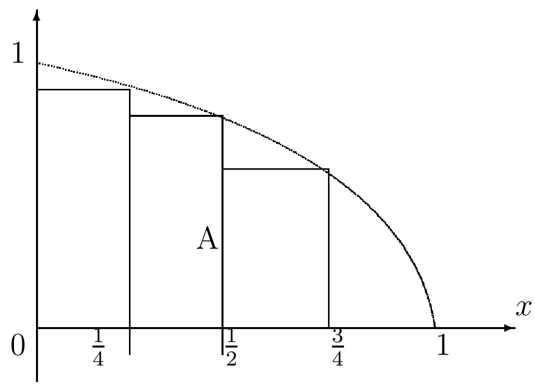

$Fig.2,\ x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$

The upper sum is

$$A \approx f(x_0)\frac{1}{4} + f(x_1)\frac{1}{4} + f(x_2)\frac{1}{4} + f(x_3)\frac{1}{4} + f(1)\frac{1}{4}$$

$$1 * 1/4 + 15/16\ 1/4, 3/4\ 1/4 + 7/161/4 = 25/12 = 0.78125$$

Also, we can approximate the area $A$ by lower sums, the rectangles lie in area $A$ (see Fig. 3)




$$Fig.3,\ x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$$

Thus, the lower sum is

$$\begin{aligned} A &\approx f(x_1)\frac{1}{4} + f(x_2)\frac{1}{4} + f(x_3)\frac{1}{4} + f(x_4)\frac{1}{4} + f(1)\frac{1}{4} \\ &\frac{15}{16} * \ \frac{1}{4} + \frac{3}{4} * \ \frac{1}{4} + \frac{7}{16} * \ \frac{1}{4} = 0.53125 \end{aligned}$$

Let us consider other way of approximation of the area $A$ by choosing mid-points of the subintervals

$$[0, 1/4],\ [1/4, 1/2],\ [1/2, 3/4],\ [3/4, 1], \qquad (n = 4).$$

The mid-points are

$$x_i^* = \frac{1}{8} + \frac{i}{4}, \qquad i = 1, 2, 3, 4$$

Now, we approximate the area $A$ by the sum (see Fig. 4)

$$\begin{aligned} A &\approx f(x_1^*)\frac{1}{4} + f(x_2^*)\frac{1}{4} + f(x_3^*)\frac{1}{4} + f(x_4^*)\frac{1}{4} = \\ &\frac{63}{64} * \frac{1}{4} + \frac{55}{64} * \frac{1}{4} + \frac{30}{64} * \frac{1}{4} + \frac{15}{64} * \frac{1}{4} = 0.671875 \end{aligned}$$

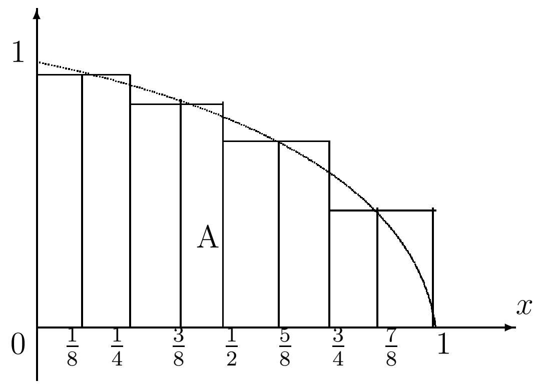



Fig. 4

Here, the points

$$x_0 = 0,\ x_1^* = \frac{1}{8},\ x_1 = \frac{1}{4},\ x_2^* = \frac{3}{8},$$
$$x_2 = \frac{1}{2},\ x_3^* = \frac{5}{8},\ x_3 = \frac{3}{4},\ x_4^* = \frac{7}{8},\ x_4 = 1$$

We shall extend the mid-point rule to a function $f(x)$ defined on an interval $[a, b]$. So, we divide the interval $[a, b]$ in $n$ equal subintervals of the length $\Delta x = \dfrac{b-a}{n}$ (see Fig.5)

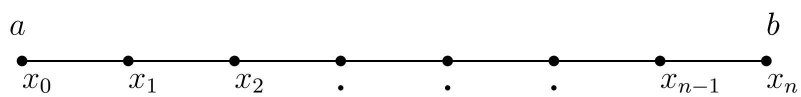



Fig.5 Partition of the interval $[a, b]$

$$[a, x_1],\ [x_1, x_2],\ [x_2, x_3], ..., [x_{n-1}, b],$$

where

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ ..., x_n = a + n\Delta x = b,$$

Then, at the mid-points

$$x_1^* = \frac{x_0 + x_1}{2},\quad x_2^* = \frac{x_1 + x_2}{2}, ..., x_n^* = \frac{x_{n-1} + x_n}{2}$$

the approximate value of the area is

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x_3 + ... + f(x_n^*)\Delta x \tag{1.1}$$

## 1.2 Sigma Notation

We shall use sigma notation for the sum of $n$ values $a_1, a_2, a_3, ..., a_n$, so we write

$$a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$$

For example, when $n = 4$ and $a_1 = 1^2,\ a_2 = 2^2,\ a_3 = 3^3,\ a_4 = 4^2$, we have

$$1^2 + 2^2 + 3^2 + 4^2 = \sum_{k=1}^{4} k^2 = 30$$

In sigma notation, formula (1.1) is

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Later, we shall use the following formulae

$$\begin{aligned} \sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= 1^3 + 2^3 + 3^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2 \end{aligned} \tag{1.2}$$

We note that

$$\sum_{k=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n$$

Also, we shall use the following additive properties of the sum

$$\begin{aligned} \sum_{k=1}^{n} [a_k \pm b_k] &= \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k \\ \sum_{k=1}^{n} c\, a_k &= c \sum_{k=1}^{n} a_k \end{aligned} \tag{1.3}$$

Let us come back to the example 1. In the example

$$f(x) = 1 - x^2, \qquad [a, b] = [0, 1].$$

We consider $n$ points

$$x_0 = 0, x_1 = \frac{1}{,}\ x_2 = \frac{2}{n},\ x_3 = \frac{3}{n}, ..., x_{n-1} = \frac{n-1}{n},\ x_n = \frac{n}{n}$$

for any natural $n$.
Then, the upper approximating sum of the area $A$ is:

$$\begin{aligned} A &\approx \sum_{k=0}^{n} f(x_k)\Delta x = \sum_{k=0)}^{n} [1-(\frac{k}{n})^2]\frac{1}{n} = \\ &= \sum_{k=0}^{n} 1\frac{1}{n} - \sum_{k=0}^{n} (\frac{k}{n})^2 \frac{1}{n} = \\ &= \frac{n+1}{n} - \frac{1}{n^3}\sum_{k=0}^{n} k^2 = \\ &= 1 + \frac{1}{n} - \frac{n(n+1)(2n+1)}{6n^3} \end{aligned}$$

Hence, the exact value of the area is the limit of the expression

$$1 + \frac{1}{n} - \frac{n(n+1)(2n+1)}{6n^3} \to 1 - \frac{1}{3} = \frac{2}{3}$$

when $n \to \infty$
In symbols, we write

$$A = \lim_{n->\infty} [1 + \frac{1}{n} - \frac{n(n+1)(2n+1)}{6n^3}] = \frac{2}{3}$$

## 1.3 Riemann Sums

In order to define Riemann Integral (Definite Integral), we begin with Riemann sums of a function $f(x)$ given for $x$ in the interval $x \in [a, b]$.
**Regular Partition**. We choose points $\{x_0, x_1, x_2, x_3, ...., x_{n-1}, x_n\} \in [a, b]$ which satisfy the inequalities

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n_1} < x_n = b$$

The set $P = \{x_0, x_1, x_2, x_3, ...., x_{n-1}, x_n\}$ is called partition of $[a, b]$. Thus, the partition divides $[a, b]$ into $n$ closed subintervals

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3], ..., [x_{n-1}, x_n].$$

The length of each subinterval, we denote by

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1, \quad ... \quad \Delta x_n = x_n - x_{n-1}$$

The partition is called regular if $\Delta x_k$ tends to zero when the number of points $n$ tends to infinity, that is

$$\Delta x_k \to 0 \qquad \text{when} \quad n \to \infty, \quad \text{for all } \ k = 1, 2, 3, ...,$$

In each subinterval $[x_{k-1}, x_k]$, $k = 1, 2, ..., n$, we choose a point $c_k$, so that $c_k \in [x_{k-1}, x_k]$.

We define Riemann sums of the function $f(x)$ in the interval $[a, b]$ on the regular partition $P$ as follows:

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k \tag{1.4}$$

## 1.4 Riemann Integral

Riemann Integral (Definite Integral) is considered as the limit of the Riemann sums. This, we state in the following definition:

**Definition 1.1** *Consider a function $f(x)$ in the interval $[a, b]$. If there exits limit of the Riemann sums*

$$\lim_{n->\infty} S_n = \lim_{n->\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = I(f)$$

*and the limit is the same for every regular partition $P$ and for every choice of the points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, ..., n$, then the function $f(x)$ is integrable in $[a, b]$ and the value of the integral equals to limit $I(f)$.*
*In symbols, we write*

$$\int_a^b f(x)\, dx = I(f).$$

We note that in the example the function $f(x) = 1 - x^2$ in $[0, 1]$ is continuous, therefore the integral

$$\int_0^1 (1 - x^2) dx$$

exists and its value $I(f) = \dfrac{2}{3}$, is obtained as the limit of Riemann sums on the uniform partition.

Let us state the theorem on existence of the Riemann integral.

**Theorem 1.1** *Every continuous function $f(x)$ on the closed interval $[a, b]$ is integrable. That is the Riemann integral*

$$\int_a^b f(x)\, dx$$

*exists.*

**Example 1.2** *Use the Riemann sums to evaluate the integral*

$$\int_1^2 (1 + 2x + 3x^2) dx$$

We observe hat the function $f(x) = 1 + 2x + 3x^2$ is continuous in the interval $[1, 2]$. Thus, by the theorem, the Riemann integral

$$\int_1^2 (1 + 2x + 3x^2)dx$$

exists. Therefore, the value $I(f)$ of the integral is the same for any choice of regular partition of $[1, 2]$. To evaluate the integral, we can choose the simplest uniform partition of $[1, 2]$ by the points

$$x_0 = 1, x_1 = 1 + \frac{1}{n},\ x_2 = 1 + \frac{2}{n},\ x_3 = 1 + \frac{3}{n}, \cdots, x_n = 1 + \frac{n}{n}$$

with $\Delta x_k = x_k - x_{k-1} = \dfrac{1}{n}$ and with the points $c_k = x_k,\ k = 1, 2, ..., n$
Then, the Riemann sum of $f(x)$ is

$$\begin{aligned} S_n = \sum_{k=1}^{n} f(x_k)\Delta x_k &= \sum_{k=1}^{n} [1 + 2x_k + 3(x_k)^2]\Delta x_k \\ &= \sum_{k=1}^{n} [1 + 2(1 + \frac{k}{n}) + 3(1 + \frac{k}{n})^2)]\frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{6}{n} + \sum_{k=1}^{n} \frac{2k}{n^2} + \sum_{k=1}^{n} \frac{6k}{n^2} + \sum_{k=1}^{n} \frac{3k^2}{n^3} \\ &= \frac{6}{n}\sum_{k=1}^{n} 1 + \frac{8}{n^2}\sum_{k=1}^{n} k + \frac{3}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{6n}{n} + \frac{8n(n+1)}{2n^2} + \frac{3n(n+1)(2n+1)}{6n^3} \\ &= 6 + 4(1 + \frac{1}{n}) + \frac{1}{2}(1 + \frac{1}{n})(2 + \frac{1}{n}) \end{aligned}$$

Then, we find the limit of the Riemann sum

$$\lim_{n->\infty} S_n = \lim_{n->\infty} [6 + 4(1 + \frac{1}{n}) + \frac{1}{2}(1 + \frac{1}{n})(2 + \frac{1}{n})] = 11$$

Hence, the integral

$$\int_1^2 (1 + 2x + 3x^2)dx = 11$$

There are discontinuous functions which are not integrable. Consider the example

**Example 1.3** *Dirichlet's function*

$$f(x) = \begin{cases} 1 & x \text{ isrational} \\ \\ 0 & x \text{ isirrational} \end{cases}$$

*is not integrable on the interval* $[0, 1]$.

Indeed, consider a regular partition of $[0,1]$. Let us choose rational points $c_k \in [x_{k-1}, x_k],\ k = 1, 2, ..., n$. Then, the Riemann sum of $f(x)$ is

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} 1\ \Delta x_k = 1 = I(f).$$

On the other hand, choosing irrational points $c_k \in [x_{k-1}, x_k],\ k = 1, 2, ..., n$ the Riemann sum is

$$\overline{S}_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} 0\ \Delta x_k = 0 = \overline{I}(f).$$

Thus, $I(f) = 1 \neq 0 = \overline{I}(f)$. So, the limit of the Riemann sum depends on a choice of the points $c_k \in [x_{k-1}, x_k]$. Therefore, by the definition, Riemann integral of Dirichlet's function does not exists in the interval $[0,1]$. However, some of discontinuous functions are integrable in an interval $[a,b]$.

**Example 1.4** *Consider the function (see Fig. 6)*

$$f(x) = \begin{cases} -1 & for \quad 0 \leq x < 1 \\ 1 & for \quad 1 \leq x \leq 2 \end{cases}$$

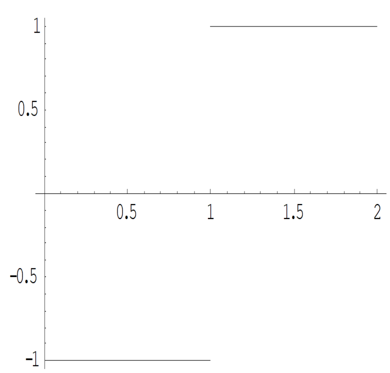



Fig. 6 Graph $f(x)$

Of course, the Riemann integral exists since the area between $x$ -axis and the graph of the function is equal 2. However, the integral

$$\int_0^2 f(x)dx = 0.$$

To show this, we follow the definition. Let us consider the Riemann sums of $f(x)$ on the intervals $[0,1]$ and $[1,2]$. First, we find the Riemann sums on the interval $[0,1]$

$$S_n^{(1)} = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}(-1)\Delta x_k = -1$$

Next, we find the Riemann sums on the interval $[1,2]$

$$S_n^{(2)} = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k-1}^{n}(-1)\Delta x_k = 1$$

for any partition $P$ of the intervals and any choice of the points $c_k$, $k = 1, 2, ..., n$.
Thus, the limit

$$\lim_{n->\infty} S_n^{(1)}(-1) + \lim_{n->\infty} S_n^{(2)} = \lim_{n->\infty}(-1) + \lim_{n->\infty} 1 = -1 + 1 = 0.$$

Hence, the function $f(x)$ is integrable and the value of the integral is equal to zero. However, the area between $x$-axis and the graph of the function $f(x)$ is equal to 2.

## 1.5 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be two integrable functions in the interval $[a,b]$, so that the integrals

$$\int_a^b f(x)dx, \quad \int_a^b g(x)dx$$

exist.
Following the definition, one can show the following equalities for definite integrals:
Additive property with respect to the integrand

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \tag{1.5}$$

Additive property with respect to the interval of integration

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad for\ any\ \ c\ \ a \leq c \leq b \tag{1.6}$$

For any constant $K$

$$\int_a^b Kf(x)dx = K\int_a^b f(x)dx \tag{1.7}$$

Reversing the direction of integration

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{1.8}$$

Let prove additive property (1.5). By the assumption, the integrals

$$\int_a^b f(x)dx, \quad \int_a^b g(x)dx$$

exist.
This means the values of the integrals are limits of the Riemann sums

$$\int_a^b f(x)dx = \lim_{n->\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k, \qquad \int_a^b g(x)dx = \lim_{n->\infty} \sum_{k=1}^{n} g(c_k)\Delta x_k.$$

Thus, we have

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \lim_{n->\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k + \lim_{n->\infty} \sum_{k=1}^{n} g(c_k)\Delta x_k =$$

$$\lim_{n->\infty} \sum_{k=1}^{n} [f(c_k) + g(c_k)]\Delta x_k = \int_a^b [f(x) + g(x)]dx$$

We note that in the above equalities, we have used the additive property of limits, that is, the limit of a sum equals the sum of the limits.
Also, by the definition, one can show the following inequalities for definite integrals:
For a non-negative and integrable function $f(x) \geq 0$ in the interval $[a, b]$, the integral

$$\int_a^b f(x)dx \geq 0. \tag{1.9}$$

Let $f(x)$ and $|f(x)|$ be a integrable functions in $[a, b]$, then

$$\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx \tag{1.10}$$

Let $f(x)$ and $g(x)$ be integrable functions in the interval $[a, b]$ and let $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

If $f(x)$ is an integrable function in the interval $[a, b]$ and if $f(x)$ is lower bonded by a number $m$ and upper bounded a number $M$, that is

$$m \leq f(x) \leq M \quad \text{for all} \quad x \in [a, b],$$

then

$$m(b-a) \leq \int_a^b f(x)dx| \leq M(b-a) \tag{1.11}$$

Let us prove the first of above inequalities. By the assumption $f(x) \geq 0$ is integrable in the interval $[0, 1]$. Then, the limit of the Riemann sums

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k \qquad n- > \infty,$$

exists and this limit of the sum of non-negative terms $f(c_k)\Delta x_k \geq 0, \ k = 1, 2, ..., n$, is non-negative, that is

$$0 \leq \lim_{n->\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx$$

We note, by property (1.9), that the area between the graph of $f(x),\ a \in [a, b]$ and x-axis is

$$A = \int_a^b |f(x)|\, dx$$

**Average of** $f(x)$. Let $f(x)$ be a continuous function in interval $[a, b]$. Then, the integral

$$\int_a^b f(x)\, dx$$

exists.
The average value of $f(x)$ in $[a, b]$ is given by the formula

$$Average(f) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

We note that the average value of $f(x)$ is the limit of arithmetic averages

$$\frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n}$$

of the values $f(x_1, f(x_2), ..., f(x_n)$ at points $x_1, x_2, x_3, ..., x_n$ in interval $[a, b]$.
Indeed, we consider the uniform partition of the interval $[a, b]$

$$x_k = a + k\Delta x,\ k = 1, 2, ..., n,\ \Delta x = \frac{b-a}{n}$$

and the Riemann sums

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f(x_k)\Delta x = \\ &= (b-a) \sum_{k=1}^{n} \frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n} \end{aligned}$$

Hence in the limit of arithmetic averages

$$\begin{aligned} \lim_{n->\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} &= \\ &= \frac{1}{b-a} \lim_{n>\infty} S_n \\ &= \frac{1}{b-a} \int_a^b f(x)\, dx \\ &= Average(f) \end{aligned}$$

The average of a continuous function $f(x)$ in a closed interval $[a, b]$ implies the following First Mean Value Theorem for integrals:

**Theorem 1.2** *If a function $f(x)$ is continuous in the closed interval $[a, b]$, then, there exists a point $\xi \in (a, b)$ in the open interval, such that*

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

*for all $x \in [a, b]$.*

Indeed, by the Weierstrass theorem, a function which is continuous in the closed interval $[a, b]$ attains its minimum value and maximum value, that is

$$m = \min_{a \le x \le b} f(x) \le f(x) \le \max_{a \le x \le b} f(x) = M$$

Also, the other Weierstrass theorem, a continuous function in a closed interval $[a, b]$ attains any value between $m$ and $M$, that is, for any number $y$, $m \le y \le M$, there is an argument $\xi \in [a, b]$ such that

$$y = f(\xi)$$

Because, the average value $Average(f)$ lies between its minimum $m$ and maximum $M$, $m \le Average(f) \le M$, therefore, by the Weierstrass theorem, there exists $\xi \in [a, b]$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

for all $x \in [a, b]$.

**Example 1.5** *Evaluate the average value of the function*

$$f(x) = \sin \pi x, \qquad 0 \le x \le \pi,$$

*in the interval* $[0, \pi]$

By the formula

$$Average(f) = \frac{1}{b-a} \int_a^b f(x)\, dx = \frac{1}{\pi} \int_0^\pi \sin x\, dx = -\frac{1}{\pi} \cos x|_0^\pi = \frac{2}{\pi}$$

**Exercises**

Question 1.

Use the definition to evaluate the following integrals as the limit of Riemann sums

(a)

$$\int_0^2 (4 - x^2) dx$$

(b)

$$\int_0^2 f(x) dx,$$

where

$$f(x) = \begin{cases} 1 & for \quad 0 \le x < 1 \\ 2x & for \quad 1 \le x \le 2 \end{cases}$$

Question 2.

(a) Use the definition to show that

$$\int_0^1 (ax^2 + bx + c)dx = \frac{1}{3}a + \frac{1}{2}b + c$$

(b) Evaluate the integral

$$\int_0^1 (3x^2 + 2x + 1)dx$$

(c) Evaluate the evarage value of the function $f(x) = x^3 + 3x^2 + 5$ in the interval $[0, 2]$.

## 1.6 Fundamental Theorems of Calculus

The relations between indefinite and definite integrals have been stated in the form of Fundamental Theorems of Calculus.

**First Fundamental Theorem of Calculus**. Assume that $f(x)$ is a continuous function in the closed interval $[a, b]$ and let $F(x)$ be an antiderivative to $f(x)$ in the interval $[a, b]$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

**Proof.** By the assumption $f(x)$ is continuous in $[a, b]$. By the theorem on existence there exists the integral

$$\int_a^b f(x)\, dx$$

Let $F(x)$ be the antiderivative to $f(x)$ in $[a, b]$ and let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a regular partition of $[a, b]$ by points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Then, by the Mean Value Theorem

$$F(x_k) - F(x_{k-1}) = F^{'}(c_k)(x_k - x_{k-1}) = F^{'}(c_k)\Delta x_k.$$

for any choice of the points $c_k \in [x_{k-1}, x_k],\ k = 1, 2, ..., n.$

Hence

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} F^{'}(c_k)\Delta_k$$

But, we have

$$\begin{aligned}\sum_{k=1}^{n}[F(x_k)-F(x_{k-1}) &= +F(x_1)-F(x_0)\\ &\quad +F(x_2)-F(x_1)\\ &\quad +F(x_3)-F(x_2)\\ &\quad \dots\dots\dots\dots\\ &\quad +F(x_{n-1})-F(x_{n-2})\\ &\quad +F(x_n)-F(x_{n-1})\\ &\quad =F(x_n)-F(x_0)\\ &\quad =F(b)-F(a)\end{aligned}$$

Hence for $F^{'}(c_k)=f(c_k),\ k=1,2,...n$, the Riemann sums

$$S_n=\sum_{k=1}^{n} f(c_k)\Delta x_k=F(b)-F(a)$$

Because $f(x)$ is integrable function in $[a,b]$, therefore taking limit of both sides, we obtain

$$\lim_{n->\infty} S_n=F(b)-F(a)=\int_a^b f(x)\ dx$$

This ends proof of the First Theorem of Calculus.

**Example 1.6** *Evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \cos\ x\ dx$$

We note that the function $f(x)=\cos\ x$ (see Fig 7) is contiguous for all real $x$ and its antiderivative $F(x)=\sin\ x$. By the theorem

$$\int_0^{\frac{\pi}{2}} \cos\ x\ dx=F(\frac{\pi}{2})-F(0)=\sin\frac{\pi}{2}-\sin\ 0=1-0=1$$

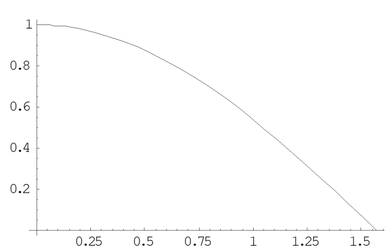



Fig. 7 $f(x)=\cos\ x,\quad x\in[0,\frac{\pi}{2}]$

The area below the graph is equal to 1.

**Theorem 1.3 Second Fundamental Theorem of Calculus** *If function $f(x)$ is continuous in the closed interval $[a, b]$, then the function*

$$F(x) = \int_a^x f(t)\ dt$$

*is continuous and differentiable and $F^{'}(x) = f(x)$ for all $x \in [a, b]$.*

**Proof.** Using the definition of derivative, we find

$$F^{'}(x) = \lim_{h->0} \frac{F(x+h) - F(x)}{h} \quad = \quad \lim_{h->0}[\frac{1}{h}\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt]$$
$$\lim_{h->0} \frac{1}{h}\int_x^{x+h} f(t)\ dt = \lim_{h->0} f(c_h)$$

for a point $c_h \in [x, x+h]$, where $f(c_h)$ is the average value of $f(x)$ in the interval $[x, x+h]$. On the other hand, the average value $f(c_h) -> f(x)$, when $h -> 0$. Thus, in the limit

$$F^{'}(x) = \lim_{h->0} f(c_h) = f(x)$$

**Example 1.7** *Using the Second Fundamental Theorem, evaluate the derivative of the function*

$$F(x) = \int_1^x (t^2 + t + 1)\ dt$$

*at the point $x = 4$.*

We note that $f(x) = x^2 + x + 1$ is continuous function for all real $x$. Thus, the assumptions of the theorem hold. By the thesis

$$F^{'}(x) = f(x) = x^2 + x + 1.$$

At the point $x = 4$, we compute $F^{'}(4) = f(4) = 4^2 + 4 + 1 = 21$.

**Example 1.8** *Evaluate the integral*

$$\int_0^1 \sin \pi x\ dx$$

The antiderivative to $f(x) = \sin \pi x$ is $F(x) = -\frac{1}{\pi}\cos \pi x$. By the thesis of the theorem

$$\int_0^1 \sin \pi x\ dx = F(1) - F(0) = -\frac{1}{\pi}[\cos \pi - \cos 0] = -\frac{1}{\pi}[-1 - 1] = \frac{2}{\pi}$$

We note that the area between the graph of the function $\sin \sin \pi x$ and x-axis is equal to $\frac{2}{\pi}$

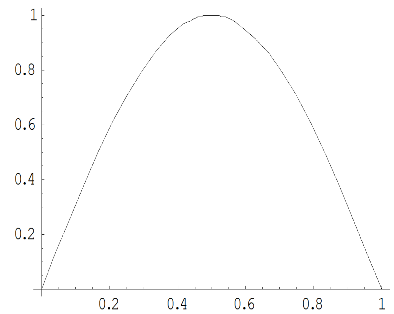



Fig. 8. $f(x) = \sin \pi x$.

## 1.7 Methods of Integration

In evaluation of Riemann integrals, we use extensively the Fundamental Theorem of Calculus.

**Theorem 1.4** *Let $F(x)$ be an antiderivative to a continuous function $f(x)$ in the interval $[a, b]$. Then, the integral*

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)$$

Thus, the methods of evaluation of Riemann integrals are the same as the methods for indefinite integrals, for finding an antiderivative.
Let us begin with the method by substitution

### 1.7.1 Method by Substitution.

Let $f(x)$ be a continuous function in $[a, b]$ and let $x = \phi(t)$ be a differentiable function in the interval $t \in [\alpha, \beta]$, with the values in $[a, b]$, that is

$$a = \phi(\alpha) \leq \phi(t) \leq \phi(\beta) = b$$

for $t \in [\alpha, \beta]$.
There exists an antiderivative $F(x)$ for the continuous function $f(x)$ in $[a, b]$. So, we consider the composed function

$$\Phi(t) = F(\phi(t)), \qquad x = \phi(t) \quad t \in [\alpha, \beta]$$

The derivative

$$\frac{d\Phi(t)}{dt} = \frac{dF(\phi(t))}{dx}\frac{d\phi(t)}{dt}, \qquad t \in [\alpha, \beta]$$

Because $\dfrac{dF(\phi(t))}{dx} = f(\phi(t))$, therefore, integrating both sides, we have

$$\Phi(\beta) - \Phi(\alpha) = \int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)\, dt$$

But $\Phi(\alpha) = F(\phi(\alpha)) = F(b)$ and $\Phi(\beta) = F(\phi(\beta)) = F(a)$
Hence, we obtain the following formula of integrating by substitution

$$\int_{a}^{b} f(x)\, dx = \int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)\, dt \qquad x = \phi(t) \tag{1.12}$$

We shall apply extensively the formula, when integrating trigonometric and rational functions.
Now, we shall consider some Riemann integrals of the following form:

$$\int_{a}^{b} f(g(x))g^{'}(x)\, dx$$

in which the functions $f(g)$ and $g(x)$ are identified.
Then, we substitute

$$u = g(x), \qquad du = g^{'}(x)\, dx.$$

to the integral for $a \leq x \leq b, \quad$ and $\quad g(a) \leq u \leq g(b)$
Thus, we obtain the following formula of integration by substitution

$$\int_{a}^{b} f(g(x))g^{'}(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du \tag{1.13}$$

**Example 1.9** *Evaluate the integral*

$$\int_{1}^{3} \frac{2x}{1+x^2}\, dx$$

**Solution** *We identify the functions*

$$g(x) = 1 + x^2, \quad g^{'}(x) = 2x, \quad f(g(x)) = \frac{1}{g(x)}, \quad 1 \leq x \leq 3$$

*Let us substitute*

$$u = x^2 + 1, \quad du = 2x\, dx$$

*The range of* $u:\ \ g(a) = 2 \leq u \leq 10 = g(b)$ *for* $1 \leq x \leq 3.$
*We find*

$$f(u) = \frac{1}{u}, \qquad du = g^{'}(x)\, dx = 2x\, dx, \quad g(1) = 2, \quad g(3) = 10$$

*By formula (2.24), we evaluate*

$$\int_{1}^{3} \frac{2x}{1+x^2}\, dx = \int_{2}^{10} \frac{1}{u}\, du == \ln 10 - \ln 2 = \ln 5$$

**Example 1.10** *Evaluate the integral*

$$\int_1^2 \frac{2x+1}{x^2+x+1}\, dx$$

**Solution** *We identify the functions*

$$g(x) = x^2 + x + 1, \quad g^{'}(x) = 2x+1, \quad f(g(x)) = \frac{1}{g(x)},, \quad 1 \le x \le 2$$

*Let us substitute*

$$u = g(x) = x^2 + x + 1, \qquad du = (2x+1)\, dx.$$

*Then, we find*

$$f(u) = \frac{1}{u}, \qquad du = g^{'}(x)\, dx = (2x+1)\, dx, \quad g(1) = 3, \qquad g(2) = 7$$

*By formula (2.24), we evaluate*

$$\int_1^2 \frac{2x+1}{x^2+x+1}\, dx = \int_3^7 \frac{1}{u}\, du = \ln\ |u||_3^7 = \ln 7 - \ln 3 = \ln\frac{7}{3}$$

**Example 1.11** *Evaluate the integral*

$$\int_0^1 2x\sqrt{1+x^2}\, dx$$

**Solution**. We identify the functions

$$g(x) = 1 + x^2, \quad g^{'}(x) = 2x, \quad f(g(x)) = \sqrt{g(x)}, \quad 0 \le x \le 1$$

Let us substitute

$$u = x^2 + 1, \quad du = 2x\, dx$$

Then, we find

$$f(u) = \sqrt{u}, \quad g(0) = 1, \quad g(1) = 2$$

By formula (2.24), we evaluate

$$\int_0^1 2x\sqrt{1+x^2}\, dx = \int_1^2 \sqrt{u}\, du = \int_1^2 u^{\frac{1}{2}}\, du = [\frac{u^{1+\frac{1}{2}}}{1+\frac{1}{2}}]|_1^2 = \frac{2}{3}[2\sqrt{2} - 1]$$

**Example 1.12** *For a given differentiable function $f(x)$, in $[a, b]$, find the integral*

$$\int_a^b \frac{f^{'}(x)}{f(x)}\, dx$$

**Solution** We note that the logarithmic derivative

$$[\ln|f(x)|]^{'} = \frac{f^{'}(x)}{f(x)}$$

Thus, the indefinite integral

$$\int \frac{f^{'}(x)}{f(x)}\, dx = \ln|f(x)| + C$$

Hence, the antiderivative to the integrand $\frac{f^{'}(x)}{f(x)}$ is $F(x) = \ln|f(x)|$.
By the Fundamental Theorem of Calculus

$$\int_a^b \frac{f^{'}(x)}{f(x)}\, dx = \ln|f(x)|_a^b = \ln f(b) - \ln f(a) = \ln\frac{f(b)}{f(a)}$$

The same formula, we can obtain by the method of substitution. Indeed, we identify the functions

$$g(x) = f(x), \quad g^{'}(x) = f^{'}(x), \qquad f(g(x)) = \frac{1}{g(x)}, \ , \quad a \le x \le b$$

Let us substitute

$$u = g(x) = f(x).$$

Then, we find

$$f(u) = \frac{1}{u}, \qquad du = g^{'}(x)\, dx = f^{'}(x)\, dx, \quad g(a) = f(a), \qquad g(b) = f(b)$$

By formula (2.24), we evaluate

$$\int_a^b \frac{f^{'}(x)}{f(x)}\, dx = \int_{f(a)}^{f(b)} \frac{1}{u}\, du = \ln|u||_{f(a)}^{f(b)} = \ln|f(b)| - \ln|f(a)| = \ln|\frac{f(b)}{f(a)}|$$

**Example 1.13** *Evaluate the integral*

$$\int_0^{\frac{\pi}{4}} \tan x\, dx$$

**Solution** *Let us write the integrand* $\tan x = \frac{\sin x}{\cos x}$ *in terms of* $\sin x$ *and* $\cos x$. *Then, we are to evaluate the integral*

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx$$

*We identify the functions*

$$g(x) = \cos x, \quad g^{'}(x) = -\sin x, \quad f(g(x)) = \frac{1}{g(x)}, \quad 0 \le x \le \frac{\pi}{4}$$

*Let us substitute*

$$u = g(x) = \cos x, \qquad g^{'}(x) = -\sin x$$

*Then, we find*

$$f(u) = \frac{1}{u}, \quad du = -\sin x \; dx,$$

$$g(0) = 1, \quad g(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

*By formula (2.24), we evaluate*

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \; dx &= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \; du \\ &= -\ln|u|_1^{\frac{1}{\sqrt{2}}} = -(\ln\frac{1}{\sqrt{2}} - \ln 1) = -\ln\frac{1}{\sqrt{2}} = \frac{\ln 2}{2} \end{aligned}$$

**Example 1.14** *For a given function $f(x)$ in $[a, b]$, find the integral*

$$\int_a^b x f(x^2) \; dx$$

*Find the integral*

$$\int_0^1 x e^{x^2} \; dx, \quad 0 \le x \le 1$$

**Solution** *We identify the functions*

$$u = g(x) = x^2, \quad g^{'}(x) = 2x, \quad du = 2x \; dx$$

*Then, we have*

$$\int_a^b x f(x^2) \; dx = \frac{1}{2}\int_a^b 2x f(x^2) \; dx = \frac{1}{2}\int_{g(a)}^{g(b)} f(u) \; du$$

*Hence, for $f(x) = e^x$, we have $f(x^2) = e^{x^2}$, $g(0) = 0$, $g(1) = 1$ Then, the integral*

$$\int_0^1 x e^{x^2} \; dx = \frac{1}{2}\int_0^1 e^u \; du = \frac{1}{2} e^u|_0^1 = \frac{1}{2}(e-1)$$

### 1.7.2 Integration by Parts

Let $u(x)$ and $v(x)$ be two differentiable function in the interval $[a, b]$. Then, the derivative of the product

$$\frac{du(x)v(x)}{dx} = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}$$

or in Newton notation

$$[u(x)v(x)]^{'} = u(x)v^{'}(x) + v(x)u^{'}(x)$$

Integrating both sides from $a$ to $b$, we find

$$u(b)v(b) - u(a)v(a) = \int_a^b u(x)\frac{dv(x)}{dx}\ dx + \int_a^b v(x)\frac{du(x)}{dx}\ dx$$

Hence, we have the formula of Integration by Parts

$$\int_a^b u(x)\frac{dv(x)}{dx}dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)\frac{du(x)}{dx}dx \qquad (1.14)$$

or

$$\int_a^b u(x)v^{'}(x)\ dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u^{'}(x)\ dx \qquad (1.15)$$

**Example 1.15** *Evaluate the integral*

$$\int_0^{\pi} x \sin\ x\ dx$$

**Solution**. Let us denote by

$$u = x,\ \ u^{'} = 1,$$

and by

$$v^{'} = x, \quad v = \int \sin\ x\ dx = -\cos\ x$$

By formula (1.15), we evaluate

$$\begin{aligned} \int_0^{\pi} x \sin\ x\ dx &= -x \cos\ x|_0^{\pi} - \int_0^{\pi} (-\cos\ x)\ dx \\ &= -(\pi \cos\ \pi - 0 \cos\ 0\ ) - \int_0^{\pi} (-cos\ x\ dx \\ &= \pi + \sin\ x\ |_0^{\pi} = \pi + (\sin \pi - \sin\ 0) = \pi \end{aligned}$$

**Example 1.16** *Evaluate the integral*

$$\int_0^1 x \ln(1+x)\ dx$$

**Solution**. Let us denote by

$$u = \ln(1+x),\ \ u^{'} = \frac{1}{1+x},$$

and

$$v' = x, \quad v = \int x\ dx = \frac{x^2}{2}$$

By formula (1.15), we evaluate

$$\begin{aligned}\int_0^1 x\ln(1+x)\,dx &= \frac{1}{2}x^2\ln(1+x)|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x}\,dx \\ &= \frac{1}{2}(\ln 2 - \ln 1) - \frac{1}{2}\int_0^1 (x-1+\frac{1}{1+x})\,dx \\ &= \frac{\ln 2}{2} - \frac{1}{2}[\frac{x^2}{2} - x + \ln(1+x)]_0^1 \\ &= \frac{\ln 2}{2} - \frac{1}{2}(\frac{1}{2} - 1 + \ln 2) = \frac{1}{4}\end{aligned}$$

**Example 1.17** *Evaluate the integral*

$$\int_0^1 xe^x\,dx$$

**Solution**. Let us denote by

$$u = x, \quad u^{'} = 1,$$

and

$$v' = e^x, \quad v = \int e^x\,dx = e^x$$

By formula (1.15), we evaluate

$$\begin{aligned}\int_1^2 xe^x\,dx &= xe^x|_1^2 - \int_1^2 e^x\,dx \\ &= (2e^2 - e) - (e^2 - e) = e^2\end{aligned}$$

**Example 1.18** *Evaluate the integral*

$$\int_0^1 \sin^2 \pi\, x\,dx$$

**Solution**. Let us denote by

$$u = \sin\ \pi x, \quad u^{'} = \pi\cos\pi\, x,$$

and

$$v' = \sin\pi\, x, \quad v = \int \sin\pi\, x\,dx = -\frac{1}{\pi}\cos\pi\, x$$

By formula (1.15), we evaluate

$$\begin{aligned}I = \int_0^1 \sin^2\pi\, x\,dx &= -\frac{1}{\pi}\sin\pi\, x\cos\pi x|_0^1 + \int_0^1 \cos^2\pi\, x\,dx \\ &= \int_0^1 (1 - \sin^2\pi x)\,dx = 1 - I\end{aligned}$$

Hence, we find $I = \dfrac{1}{2}$ and

$$\int_0^1 \sin^2\pi\, x\,dx = \frac{1}{2}$$

### 1.7.3 Integration of Rational Functions

The general form of rational functions

$$\frac{Polynomial\ P_m(x)\ of\ degree\ m}{Polynomial\ Q_n(x)\ of\ degree\ n} = \frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}$$

We shall consider $m < n$, otherwise, the quotient

$$\frac{P_m(x)}{Q_n(x)} = \overline{P}_{m-n}(x) + \frac{R_{m-n}(x)}{Q_n(x)}$$

where $\overline{P}_{m-n}(x)$ is the polynomial of degree $m - n$ and the remainder polynomial $R_{m-n}(x)$ has degree $m - n < m$.
A special rule play the rational functions called Partial Fractions of the following form:

$$\begin{array}{lll} \frac{A}{x-\lambda}, & \frac{A}{(x-\lambda)^k}, & k = 2, 3, ... \\ \frac{Mx+N}{x^2+px+q} & \frac{Mx+N}{(x^2+px+q)^k}, & p^2 - 4q < 0. \end{array} \tag{1.16}$$

The following theorem holds:

**Theorem 1.5** *Every rational function* $(m < n)$ *possesses form of a sum of Partial Fractions. That is*

$$\begin{aligned} \frac{P_m(x)}{Q_n(x)} &= \frac{A_1}{(x-\lambda)} + \frac{A_2}{(x-\lambda)^2} + \cdots + \frac{A_k}{(x-\lambda)^k} \\ &+ \frac{M_1x+N_1}{(x^2+px+q)} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \cdots \end{aligned}$$

Therefore, integration of rational functions reduces to the integration of Partial Fractions.
Let us integrate the Partial Fractions

1.

$$\begin{aligned} \int_a^b \frac{A}{(x-\lambda)}\,dx &= A\ln|x-a||_a^b = A(\ln|b-\lambda| - \ln|a-\lambda|) \\ &= A\ln|\frac{b-\lambda}{a-\lambda}| \end{aligned}$$

**Example 1.19** *Evaluate the integral*

$$\int_0^1 \frac{x\,dx}{x^2-5x+6}$$

**Solution.** The rational function

$$\frac{x}{x^2-5x+6}$$

we present in partial fractions as follows: Factorizing the denominator $x^2 - 5x + 6 = (x-2)(x-3)$, we find

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{(A+B)x-(3A+2B)}{x^2-5x+6}$$

for every real $x \neq 2$ and $x \neq 3$
Hence, by comparison of the numerators, we have the equations for $A$ and $B$

$$A + B = 1 \qquad \text{and} \quad 3A + 2B = 0$$

The solution is $A = -2$ and $B = 3$. Thus, the integrand has the following partial fractions representation

$$\frac{x}{x^2-5x+6} = -\frac{2}{x-2} + \frac{3}{x-3}$$

Hence, we evaluate

$$\begin{aligned} \int_0^1 \frac{x\,dx}{x^2-5x+6} &= \int_0^1 \frac{3\,dx}{x-3} - \int_0^1 \frac{2\,dx}{x-2} \\ &= (3\ln|x-3| - 2\ln|x-2|)|_0^1 \\ &= 3\ln 2 - 3\ln 3 + 2\ln 2 = 5\ln 2 - 3\ln 3 = \ln\tfrac{32}{27} \end{aligned}$$

2. The second type of partial fractions

$$\int_a^b \frac{A}{(x-\lambda)^k}dx, \qquad k = 2, 3, ..., \qquad \lambda \notin [a, b]$$

we find the indefinite integral

$$\int \frac{A}{(x-\lambda)^k}dx = \frac{A}{1-k}(x-\lambda)^{1-k} + C \qquad k = 2, 3, ...,$$

Then, the integral

$$\int_a^b \frac{A}{(x-\lambda)^k}dx = \frac{A}{1-k}(x-\lambda)^{1-k}|_a^b = \frac{A}{1-k}[(b-\lambda)^{1-k} - (a-\lambda)^{1-k}$$

3. Let us integrate the third type of partial fractions

$$\int_a^b \frac{A}{x^2+px+q}dx \quad \text{or} \quad \int_a^b \frac{Ax+B}{(x^2+px+q}dx, \quad p^2 - 4q < 0,$$

**General approach**
To find the first integral, we present the denominator in the form

$$x^2 + px + q = (x + \frac{p}{2})^2 + (q - \frac{p^2}{4})$$

Then, we substitute

$$u = x + \frac{p}{2}, \;\; du = dx, \qquad r = \sqrt{q - \frac{p^2}{4}}$$

and we find the indefinite integral

$$\begin{aligned} \int \frac{A}{x^2+px+q}dx &= \int \frac{A\, du}{u^2+r^2} + C \\ &= \frac{A}{r} ArcTan \frac{u}{r} + C \\ &= \frac{A}{\sqrt{q-\frac{p^2}{4}}} ArcTan \frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C \end{aligned} \tag{1.17}$$

**Example 1.20** *Evaluate the integral*

$$\int_1^2 \frac{dx}{x^2-2x+2}$$

**Solution**. The denominator

$$x^2 - 2x + 2 = (x-1)^2 + 1$$

Then, we have $p = -2, \;\; q = 2$. Let $u = x - 1, \quad du = dx$ and for $x = 1, \; u = 0$ and for $x = 2, \; u = 1$

$$\begin{aligned} \int_1^2 \frac{dx}{x^2-2x+2} &= \int_0^1 \frac{du}{u^2+1} = ArcTanu|_0^1 \\ &= ArcTan(x-1)|_1^2 = ArcTan1 - ArcTan0 = \frac{\pi}{4} \end{aligned}$$

**General approach to the second integral**
We find the indefinite integral

$$\begin{aligned} \int \frac{Ax+B}{x^2+px+q}dx &= \frac{A}{2}\int \frac{2x+\dfrac{2B}{A}}{x^2+px+q}dx \\ &= \frac{A}{2}\int \frac{2x+p+\frac{2B}{A}-p}{x^2+px+q}dx \\ &= \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \frac{A}{2}\int \frac{\frac{2B}{A}-p}{x^2+px+q}dx \\ &= \frac{A}{2}\ln|x^2+px+q| + \frac{A}{2}\int \frac{\frac{2B}{A}-p}{x^2+px+q}dx \end{aligned}$$

The second integral in the above is in the form of the first integral, then we use formula (1.17).

**Example 1.21** *Evaluate the integral*

$$\int_0^2 \frac{3x+1}{x^2-4x+5}dx$$

**Solution** We note that it is integral in the form

$$\int_a^b \frac{Ax+B}{(x^2+px+q}dx$$

where $A = 3,\ B = 1,\ p = -4,\ q = 5$ and the discriminant $\Delta = p^2 - 4q = (-4)^2 - 4\ 5 = -4 < 0$.
Then, we evaluate

$$\begin{aligned}
\int \frac{3x+1}{x^2-4x+5}dx &= \frac{3}{2}\int \frac{2x+\frac{2}{3}}{x^2-4x+5}dx \\
&= \frac{3}{2}\int \frac{2x-4+\frac{2}{3}+4}{x^2-4x+5}dx \\
&= \frac{3}{2}\int \frac{2x-4}{x^2-4x+5}dx + \frac{3}{2}\int \frac{\frac{2}{3}+4}{x^2-4x+5}dx \\
&= \frac{3}{2}\ln|x^2-4x+5| + 7\int \frac{dx}{x^2-4x+5}
\end{aligned}$$

Now, we evaluate the integral in the above of the first form

$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = ArcTan(x-2)+C$$

Finally, we find the indefinite integral

$$\int \frac{3x+1}{x^2-4x+5}dx = \frac{3}{2}\ln|x^2-4x+5| + 7ArcTan(x-2)+C$$

Hence, the integral

$$\begin{aligned}
\int_0^2 = \frac{3x+1}{x^2-4x+5}dx &= [\frac{3}{2}\ln|x^2-4x+5| + 7ArcTan(x-2)]|_0^2 \\
&= [\frac{3}{2}\ln|2^2-4\ 2+5| + 7ArcTan(2-2)] \\
&- [\tfrac{3}{2}\ln|0^2-4\ 0+5| + 7ArcTan(0-2)] \\
&= 7ArcTan2 - \frac{3}{2}\ln\ 5
\end{aligned}$$

### 1.7.4 Integration Using Table of Integrals

Let us recall the antiderivative of some elementary functions

$$[ArcTanx]' = \frac{1}{1+x^2}, \qquad \int \frac{dx}{1+x^2} = ArcTanx + C,\ -\infty < x < \infty,$$

$$[tan\ x]' = \frac{1}{\cos^2 x} = \sec^2 x\ , \quad \int \frac{dx}{\cos^2 x} = tanx + C, \quad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2},$$

$$[ArcSinx]' = \frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = ArcSinx + C,\ -1 < x < 1$$

$$[ArcSinhx]' = \frac{1}{\sqrt{1+x^2}}, \qquad \int \frac{dx}{\sqrt{1+x^2}} = ArcSinhx + C,\ -\infty < x < \infty$$

**Example 1.22** *Evaluate the integral*

$$\int_0^a \frac{dx}{a^2+x^2}$$

**Solution.** Let us substitute $x = at,\ dx = adt$. Then, we find the indefinite integral

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}ArcTant = \frac{1}{a}ArcTan\frac{x}{a} + C, \quad a \neq 0$$

Hence, we have

$$\int_0^a \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{(-1)}t = \frac{1}{a}ArcTan\frac{x}{a}|_0^a = \frac{\pi}{4a}$$

**Example 1.23** *Evaluate the integral*

$$\int_0^b \frac{dx}{(a^2+x^2)^2}$$

**Solution**. To find the indefinite integral, we apply integration by parts to the integral

$$I_1 = \int \frac{dx}{a^2+x^2}$$

Let

$$u = \frac{1}{a^2+x^2}, \quad u' = \frac{-2x}{(a^2+x^2)^2}, \quad v = 1, \quad v' = x$$

$$I_1 = \int \frac{dx}{a^2+x^2} \quad = \quad \frac{x}{a^2+x^2} + 2\int \frac{x^2\ dx}{(a^2+x^2)^2}$$

But, the integral

$$\begin{aligned}\int \frac{x^2\ dx}{(a^2+x^2)^2} &= \int \frac{x^2+a^2-a^2\ dx}{(a^2+x^2)^2} \\ &= \int \frac{dx}{a^2+x^2} - a^2\int \frac{dx}{(a^2+x^2)^2}\end{aligned}$$

Hence, we have

$$I_1 = \; = \; \frac{x}{a^2+x^2} + 2(I_1 - a^2 I_2)$$

and

$$I_2 = \int \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a^2}\frac{x}{a^2+x^2} + \frac{1}{2a^2} I_1$$

Because

$$I_1 = \frac{1}{a} ArcTan \frac{x}{a} + C$$

therefore

$$\int \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a^2}\frac{x}{a^2+x^2} + \frac{1}{2a^2}\frac{1}{a} ArcTan \frac{x}{a} + C$$

Thus, the definite integral

$$\begin{aligned} \int_0^b \frac{dx}{(a^2+x^2)^2} &= [\frac{1}{2a^2}\frac{x}{a^2+x^2} + \frac{1}{2a^2}\frac{1}{a} ArcTan \frac{x}{a}]|_0^b \\ &= [\frac{1}{2a^2}\frac{b}{a^2+b^2} + \frac{1}{2a^2}\frac{1}{a} ArcTan \frac{b}{a} \end{aligned}$$

**Example 1.24** *Evaluate the integral*

$$\int_0^{\frac{\pi}{4}} \sec^2 bx\; dx, \qquad 0 < b < 1.$$

**Solution.** Let $u = bx,\; du = b\; dx$. We find the indefinite integral

$$\int \sec^2 bx\; dx = \frac{1}{b}\int \sec^2 u\; du = \frac{1}{b}\tan u + C = \frac{1}{b}\tan bx + C$$

Hence, we have

$$\int_0^{\frac{\pi}{4}} \sec^2 bx\; dx = \frac{1}{b}\tan bx|_0^{\frac{\pi}{4}} = \frac{1}{b}\tan\frac{\pi b}{4}$$

**Example 1.25** *Evaluate the integral*

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}}, \qquad a > 0.$$

**Solution.** Let us substitute $x = at,\; dx = adt$. Then, we find the indefinite integral

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = ArcSin\; t = ArcSin \frac{x}{a} + C, \quad a > 0$$

Hence, we have

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = ArcSin\; t|_0^1 = ArcSin \frac{x}{a}|_0^a = \frac{\pi}{2}$$

**Example 1.26** *Evaluate the integral*

$$\int_0^1 \frac{dx}{\sqrt{a^2+x^2}}, \quad a>0.$$

**Solution.** Let us substitute $x = at$, $dx = adt$. Then, we find the indefinite integral

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{dt}{\sqrt{1+t^2}} = ArcSinh\, t = ArcSinh\frac{x}{a} + C, \quad a>0$$

Hence, we have

$$\int_0^1 \frac{dx}{\sqrt{a^2+x^2}} = ArcSinh\, t|_0^1 = ArcSinh\frac{x}{a}|_0^1 = ArcSinh\frac{1}{a}$$

Let us recall that

$$\sinh\ x = \frac{1}{2}(e^x - e^{-x})$$

**Example 1.27** *Evaluate the integral*

$$\int_0^a \sqrt{a^2+x^2}\ dx.$$

**Solution.** To find the indefinite integral, we apply integration by parts

$$\begin{aligned}\int \sqrt{a^2+x^2}\ dx &= x\sqrt{a^2+x^2} - \int \frac{x^2\ dx}{\sqrt{a^2+x^2}} \\ &= x\sqrt{a^2+x^2} - \int \frac{(x^2+a^2) - a^2\ dx}{\sqrt{a^2+x^2}} \\ &= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\ dx + a^2 \int \frac{dx}{\sqrt{a^2+x^2}}\end{aligned}$$

By the example, we find

$$\int \frac{dx}{\sqrt{a^2+x^2}} = ArcSinh\frac{x}{a}$$

Hence, we have

$$I = \int \sqrt{a^2+x^2}\ dx \ = \ x\sqrt{a^2+x^2} - I + a^2 ArcSinh\frac{x}{a} + C$$

and

$$\int \sqrt{a^2+x^2}\ dx = \frac{1}{2}[x\sqrt{a^2+x^2} + a^2 ArcSinh\frac{x}{a} + C$$

Thus, the definite integral

$$\int_0^a \sqrt{a^2+x^2}\ dx = \frac{1}{2}[x\sqrt{a^2+x^2} + a^2 ArcSinh\frac{x}{a}]|_0^a = \frac{1}{2}[a^2\sqrt{2} + a^2 ArcSinh 1] \tag{1.18}$$

**Example 1.28** *Evaluate the integral*

$$\int_0^b \frac{dx}{(a^2+x^2)^n}, \qquad n \geq 2, \quad a > 0.$$

**Solution.** To find the indefinite integral, we apply integration by parts to the integral

$$\int \frac{dx}{(a^2+x^2)^n}$$

Let

$$u = \frac{1}{(a^2+x^2)}, \quad u^{'} = \frac{-2nx}{(a^2+x^2)^2}, \quad v = 1, \quad v^{'} = x$$

$$I_n = \int \frac{dx}{(a^2+x^2)^n} \quad = \quad \frac{x}{(a^2+x^2)^n} + 2n\int \frac{x^2\,dx}{(a^2+x^2)^{n+1}}$$

But, the integral

$$\begin{aligned}\int \frac{x^2\,dx}{(a^2+x^2)^{n+1}} &= \int \frac{x^2+a^2-a^2\,dx}{(a^2+x^2)^{n+1}} \\ &= \int \frac{dx}{(a^2+x^2)^n} - a^2\int \frac{dx}{(a^2+x^2)^{n+1}}\end{aligned}$$

Hence, we have

$$I_n = \int \frac{dx}{(a^2+x^2)^n} \quad = \quad \frac{x}{(a^2+x^2)^n} + 2n(I_n - a^2 I_{n+1}) \tag{1.19}$$

Thus, we get the recursive formulas

$$I_{n+1} = \frac{1}{2na^2}\frac{x}{(a^2+x^2)^n} + \frac{2n-1}{2na^2}I_n \tag{1.20}$$

Let us note that

$$I_1 = \int \frac{dx}{a^2+x^2} = \frac{1}{a}ArcTan\frac{x}{a} + C$$

and from formula (1.20), for $n = 1$, we find the integral

$$\int \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a^2}\frac{x}{a^2+x^2} + \frac{1}{2a^2}\frac{1}{a}ArcTan\frac{x}{a} + C$$

and the definite integral

$$\begin{aligned}\int_0^b \frac{dx}{(a^2+x^2)^2} &= [\frac{1}{2a^2}\frac{x}{a^2+x^2} + \frac{1}{2a^3}ArcTan\frac{x}{a}]|_0^b \\ &= \frac{1}{2a^2}\frac{b}{a^2+b^2} + \frac{1}{2a^3}ArcTan\frac{b}{a}\end{aligned}$$

### 1.7.5 Integration of the expression $\sqrt{ax^2+bx+c}$

Consider the integral

$$\int \sqrt{ax^2+bx+c}dx, \qquad a>0, \;\; \Delta = b^2-4ac<0. \tag{1.21}$$

In order to find the integral, we apply Euler's substitution

$$\sqrt{ax^2+bx+c} = t - x\sqrt{a}$$

Hence,we find

$$ax^2+bx+c = t^2 - 2tx\sqrt{a} + ax^2$$

$$bx+c = t^2-2tx\sqrt{a}, \qquad x = \frac{t^2-c}{2t\sqrt{a}+b},$$

$$\sqrt{ax^2+bx+c} = t - \frac{t^2-c}{2t\sqrt{a}+b} = \sqrt{a} = \frac{t^2\sqrt{a}+bt+c\sqrt{a}}{2t\sqrt{a}+b}$$

$$dx = 2\frac{t^2\sqrt{a}+bt+c\sqrt{a}}{(2t\sqrt{a}+b)^2}$$

In this way, we have transformed the integral (1.21) to the integral of a rational function

$$\int \sqrt{ax^2+bx+c}\,dx = 2\int (\frac{t^2\sqrt{a}+bt+c\sqrt{a}}{2t\sqrt{a}+b})(\frac{t^2\sqrt{a}+bt+c\sqrt{a}}{(2t\sqrt{a}+b)^2})dt \tag{1.22}$$

**Example 1.29** *Evaluate the integral*

$$\int_0^1 \sqrt{x^2+1}\,dx$$

We apply the Euler's substitution

$$\sqrt{x^2+1} = t - x$$

Hence,we find

$$x^2+1 = t^2-2tx+x^2$$

$$1 = t^2-2tx, \qquad x = \frac{t^2-1}{2t}, \quad t \neq 0$$

$$\sqrt{x^2+1} = t - \frac{t^2-1}{2t} = \frac{t^2+1}{2t}$$

$$dx = 2\frac{t^2+1}{(2t)^2}dt$$

In this way, we have transformed the integral (1.21) to the integral of a rational function

$$\begin{aligned}\int \sqrt{x^2+1}\,dx &= 2\int (\frac{t^2+1}{2t})(\frac{t^2+1}{4t^2})dt \\ &= 2\int (\frac{(t^2+1)^2}{8t^3}\,dt\end{aligned} \tag{1.23}$$

The rational function has the partial fraction representation

$$\frac{(t^2+1)^2}{8t^3} = \frac{1}{8t^3} + \frac{1}{4t} + (\frac{t}{8}$$

Thus, we find the integral

$$\int (\frac{(t^2+1)^2}{8t^3}\, dt = -\frac{1}{16t^2} + \frac{t^2}{16} + \frac{\ln t}{4} + C$$

We note that when $x = 0$ then $t = 1$, when $x = 1$ then $t = 1+\sqrt{2}$. So that

$$\begin{aligned} \int_0^1 \sqrt{x^2+1}\, dx &= 2\int_1^{1+\sqrt{2}} \frac{(t^2+1)^2}{8t^3}\, dt \\ &= [-\frac{1}{16t^2} + \frac{t^2}{16} + \frac{\ln t}{4}]|_1^{1+\sqrt{2}} \\ &= \frac{1}{2}(\sqrt{2} + ArcSinh(1)) = 1.14779 \end{aligned} \tag{1.24}$$

**Example 1.30** *Evaluate the integral*

$$\int_0^1 \sqrt{x^2+x+1}\, dx$$

We apply the Euler's substitution

$$\sqrt{x^2+x+1} = t - x$$

Hence,we find

$$x^2+x+1 = t^2 - 2tx + x^2$$

$$x+1 = t^2 - 2tx, \qquad x = \frac{t^2-1}{2t+1}, \quad t \neq -1$$

$$\sqrt{x^2+x+1} = t - \frac{t^2-1}{2t+1} = \frac{t^2+t+1}{2t+1}$$

$$dx = 2\frac{t^2+t+1}{(2t+1)^2}dt$$

In this way, we have transformed the integral (1.21) to the integral of a rational function

$$\begin{aligned} \int \sqrt{x^2+x+1}\, dx &= 2\int (\frac{t^2+t+1}{2t+1})(\frac{t^2+t+1}{(2t+1)^2})dt \\ &= 2\int (\frac{(t^2+t+1)^2}{(2t+1)^3}\, dt \end{aligned} \tag{1.25}$$

The rational function has the partial fraction representation

$$\frac{(t^2+t+1)^2}{(2t+1)^3} = \frac{t}{128} + \frac{9}{(2t+1)^2} + (\frac{3}{8(2t+1)}$$

Thus, we find the integral

$$\int (\frac{(t^2+t+1)^2}{(2t+1)^3}\, dt = \frac{1}{64}[-\frac{9}{(2t+1)^2} + 12\ln(1+2t) + (1+2t)^2] + C$$

We note that when $x = 0$ then $t = 1$, when $x = 1$ then $t = 1 + \sqrt{3}$. So that

$$\begin{aligned}
\int_0^1 \sqrt{x^2+x+1}\, dx &= 2\int_1^{1+\sqrt{3}} (\frac{(t^2+t+1)^2}{(t+1)^3}\, dt \\
&= \frac{1}{64}[-\frac{9}{(2t+1)^2} + 12\ln(1+2t) + (1+2t)^2]|_1^{1+\sqrt{3}} \\
&= 1.33691
\end{aligned} \tag{1.26}$$

### 1.7.6 Integration of Trigonometric Expressions

In the integration of trigonometric expressions, we often apply trigonometric identities

$$\begin{aligned}
&\sin^2 x + \cos^2 x = 1, && \sin 2x = 2\sin x\ \cos x, \\
&\cos 2x = cos^2 x - \sin^2 x, && \sin^2 x = \frac{1-\cos 2x}{2}, \\
&\cos^2 x = \frac{1+\cos 2x}{2}, && sec^2 x = \frac{1}{\cos^2 x} = 1 + \tan^2 x \\
&\sin x = \frac{2tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}, && \cos\ x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}
\end{aligned} \tag{1.27}$$

**Example 1.31** *Evaluate the integral*

$$\int_0^{\pi} \sin^2 x\ dx$$

**Solution**. By trigonometric identities (1.27), we find

$$\begin{aligned}
\int \sin^2 x\ dx &= \int (\sqrt{\frac{1-\cos 2x}{2}})^2 dx \\
&= \int \frac{1-\cos 2x}{2} dx \\
&= \frac{x}{2} - \frac{1}{2}\int \cos 2x\ dx \\
&= \frac{x}{2} - \frac{1}{4}\sin 2x + C
\end{aligned}$$

Hence, we evaluate

$$\int_0^{\pi} \sin^2 x\ dx = (\frac{x}{2} - \frac{1}{4}\sin 2x)|_0^{\pi} = \frac{\pi}{2}$$

Also, we can evaluate the integral using the formula of integration by parts

$$\int u\, v^{'}\, dx = u\, v - \int u^{'}\, v\, dx.$$

Let

$$u = \sin\ x, \quad u^{'} = \cos x, \qquad \text{and} \quad v^{'} = \sin x, \quad v = -\cos x$$

Then, we find the trigonometric identities to find

$$\begin{aligned} I = \int \sin^2 x\ dx &= -\sin x\ \cos x + \int \cos^2 x\ dx \\ &= -\sin x\ \cos x + \int (1 - \sin^2 x\ dx\ dx \\ &= -\sin x\ \cos\ x + x - I \end{aligned}$$

Solving for $I$, we obtain

$$\int \sin^2 x\ dx = \frac{1}{2}(x - \sin x\ \cos x)$$

Hence, we find

$$\int_0^{\pi} \sin^2 x\ dx = \frac{1}{2}(x - \sin x\ \cos x)|_0^{\pi} = \frac{\pi}{2}$$

**Example 1.32** *Evaluate the integral*

$$\int_0^{\pi} \sin^4 x\ dx$$

**Solution**. We apply the trigonometric identities (1.27) to find

$$\begin{aligned} \int \sin^4 x\ dx &= = \int (\sqrt{\frac{1 - \cos 2x}{2}})^4\ dx \\ &= = \int \frac{1}{4}(1 - \cos 2x)^2\ dx \\ &= \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x)\ dx \\ &= \int \frac{1}{4}[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]\ dx \\ &= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x \end{aligned}$$

Hence, we find

$$\int_0^{\pi} \sin^4 x\ = [\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x]|_0^{\pi} = \frac{3\pi}{8}$$

**Example 1.33** *Consider the indefinite integral*

$$I_n = \int \sin^n x\ dx, \qquad n \geq 2.$$

*(a) find the integral $I_0, I_1$ and $I_2$.*

*(b) Show the recursive formula*

$$I_n = \frac{1}{n}\sin^{n-1} x\ \cos x + \frac{n-1}{n} I_{n-2}, \qquad n \geq 2$$

*(c) Using the recursive formula, evaluate the integrals*

$$\int_0^{\frac{\pi}{2}} \sin^3 x\ dx \qquad \int_0^{\frac{\pi}{2}} \sin^4 x\ dx$$

**Solution**.
$(a)$ The straight foreword integration gives the integrals

$$I_0 = \int dx = x + C, \qquad I_1 = \int \sin x\ dx = -\cos x + C$$

For $n = 2$, we apply integration by part formula

$$\int u\ v^{'}\ dx = u\ v - \int u^{'}\ v\ dx$$

Let

$$u = \sin\ x, \quad u^{'} = \cos x, \qquad \text{and} \quad v^{'} = \sin x, \quad v = -\cos x$$

Then, we use the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to find

$$\begin{aligned} I = \int \sin^2 x\ dx &= -\sin x\ \cos x + \int \cos^2 x\ dx \\ &= -\sin x\ \cos x + \int (1 - \sin^2 x\ dx\ dx \\ &= -\sin x\ \cos\ x + x - I \end{aligned}$$

Solving for $I$, we obtain

$$\int \sin^2 x\ dx = \frac{1}{2}(x - \sin x\ \cos x)$$

Hence, we find

$$\int_0^{\pi} \sin^2 x\ dx = \frac{1}{2}(x - \sin x\ \cos x)|_0^{\pi} = \frac{\pi}{2}$$

To (b). Again, we apply the formula of integration by parts

$$\int u\ v^{'}\ dx = u\ v - \int u^{'}\ v\ dx.$$

Let

$$u = \sin^{n-1} x, \qquad u^{'} = (n-1)\sin^{n-2} x\ \cos x,$$

and

$$v' = \sin x, \quad v = -\cos x$$

Then, we have

$$\begin{aligned} I_n &= \int \sin^n x\ dx = -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} cos^2 x\ dx \\ &= -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2}(1-\sin^2 x)\ dx \\ &= -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2}\ dx - (n-1)\int \sin^n x\ dx \\ &= -\sin^{n-1} x\ \cos x + (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

Hence, we find the recursive formula

$$I_n = -\frac{1}{n}\sin^{n-1} x\ \cos x + \frac{n-1}{n} I_{n-2}$$

To $(c)$. In order to evaluate the integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x\ dx$$

we apply the recursive formula

$$\begin{aligned} I_3 &= -\frac{1}{3}\sin^2 x\ \cos x + \frac{2}{3}I_1 \\ &= -\frac{1}{3}\sin^2 x\ \cos x - \frac{2}{3}\cos x \end{aligned}$$

By Fundamental Theorem of Calculus, we evaluate

$$\int_0^{\frac{\pi}{2}} \sin^3 x\ dx = [-\frac{1}{3}\sin^2 x\ \cos x - \frac{2}{3}\cos x]|_0^{\frac{\pi}{2}} = \frac{2}{3}$$

Also, we evaluate the integral

$$\int_0^{\frac{\pi}{2}} \sin^4 x\ dx$$

by the recursive formula

$$\begin{aligned} I_4 &= -\frac{1}{4}\sin^3 x\ \cos x + \frac{3}{4}I_2 \\ &= -\frac{1}{4}\sin^3 x\ \cos x + \frac{3}{4}(-\frac{1}{2}\sin x\ \cos x + \frac{1}{2}I_0) \\ &= -\frac{1}{4}\sin^3 x\ \cos x - \frac{3}{8}\sin x\ \cos x + \frac{3}{8}x \end{aligned}$$

By Fundamental Theorem of Calculus, we evaluate

$$\int_0^{\frac{\pi}{2}} \sin^4 x\ dx = [-\frac{1}{4}\sin^3 x\ \cos x - \frac{3}{8}\sin x\ \cos x + \frac{3}{8}x]_0^{\frac{\pi}{2}} = \frac{3\pi}{16}$$

**Example 1.34** *Evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^5 x \, dx$$

**Solution**. We apply the method of substitution. Let

$$t = \sin x, \quad dt = \cos x \, dx,$$

Then, we find the indefinite integral

$$\begin{aligned} \int \sin^6 x \ \cos^5 x \, dx &= \int \sin^6 x \ \cos^4 x \ \cos x \, dx \\ &= \int t^6 (1 - t^2)^2 dt = \frac{t^7}{7} - \frac{2t^9}{9} + \frac{t^1 1}{11} \\ &= \frac{\sin^7 x}{7} - \frac{2 \sin^9 x}{9} + \frac{\sin^{11} x}{11} \end{aligned}$$

Hence, we evaluate

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^5 x \, dx = [\frac{\sin^7 x}{7} - \frac{2 \sin^9 x}{9} + \frac{\sin^{11} x}{11}]|_0^{\frac{\pi}{2}} = \frac{8}{693}$$

**Example 1.35** *Evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x} \, dx,$$

**Solution** We apply the substitution

$$\begin{aligned} t &= tan \frac{x}{2}, \quad dt = \frac{1}{2} \sec^2 \frac{x}{2} dx = \frac{1}{2}(1 + tan^2 \frac{x}{2}) dx, \\ dx &= \frac{2dt}{1 + t^2}, \quad \cos x = \frac{1 - tan^2 \frac{x}{2}}{1 + tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2} \end{aligned}$$

Then, we find indefinite integral

$$\begin{aligned} \int \frac{dx}{1 + \cos x} &= 2 \int \frac{dt}{(1 + t^2)(1 + \frac{1 - t^2}{1 + t^2})} = \int dt = t + C \\ &= \tan \frac{x}{2} + C \end{aligned}$$

Hence, the integral

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x} \, dx = \tan \frac{x}{2}|_0^{\frac{\pi}{2}} = 1$$

**Example 1.36** *Similarly, we evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x} \, dx,$$

**Solution** We apply the substitution

$$t = tan\frac{x}{2}, \qquad dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}(1 + tan^2\frac{x}{2})dx,$$
$$dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

Then, we find indefinite integral

$$\begin{aligned}\int \frac{dx}{1+\sin x} &= \int \frac{2\ dt}{(1+t^2)(1+\frac{2t}{1+t^2})} = \int \frac{2dt}{(1+t)^2} = -\frac{2}{1+t} \\ &= -\frac{2}{1+\tan\frac{x}{2}}\end{aligned}$$

Hence, the integral

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}\ dx = \frac{-2}{1+\tan\frac{x}{2}}|_0^{\frac{\pi}{2}} = 1$$

**Example 1.37** *Find the indefinite integral*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+b\cos x}, \qquad a > b > 0.$$

*Use the Fundamental theorem of Calculus to evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}.$$

**Solution** We apply the universal substitution

$$t = tan\frac{x}{2}, \qquad dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}(1 + tan^2\frac{x}{2})dx,$$
$$dx = \frac{2dt}{1+t^2}, \quad \cos x = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

Then, we have

$$\begin{aligned}\int \frac{dx}{a+b\cos x} &= \int \frac{2dt}{(1+t^2)(a+b\frac{1-t^2}{1+t^2})} \\ &= \int \frac{2dt}{(1+t^2)a+(1-t^2)b} \\ &= \int \frac{2dt}{(a-b)t^2+(a+b)} \\ &= \frac{2}{\sqrt{a^2-b^2}}\int \frac{du}{1+u^2} = \frac{2}{\sqrt{a^2-b^2}}ArcTanu,\end{aligned}$$

where

$$u = \sqrt{\frac{a-b}{a+b}}t, \qquad du = \sqrt{\frac{a-b}{a+b}}dt$$

In terms of the original variable $x$

$$\begin{aligned}\int \frac{dx}{a+b\cos x} &= \frac{2}{\sqrt{a^2-b^2}}ArcTanu \\ &= \frac{2}{\sqrt{a^2-b^2}}ArcTan\sqrt{\frac{a-b}{a+b}}t \\ &= \frac{2}{\sqrt{a^2-b^2}}ArcTan(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2})\end{aligned}$$

Hence, we find

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}}ArcTan(\sqrt{\frac{1}{3}}\tan\frac{x}{2})|_0^{\frac{\pi}{2}} = \frac{2}{\sqrt{3}}ArcTan\frac{1}{\sqrt{3}}$$

**Example 1.38** *Find the indefinite integral*

$$\int \frac{dx}{1+\sin x+\cos x}, \qquad x \neq (2k-1)\pi,\ -\frac{\pi}{2}+2k\pi, \quad k=0,\pm1,\pm2,...$$

*Use the Fundamental theorem of Calculus to evaluate the integral*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$$

**Solution** Let us substitute

$$\begin{aligned}&t = tan\frac{x}{2}, \qquad dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}(1+tan^2\frac{x}{2})dx, \\ &dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{2t}{1+t^2}\end{aligned}$$

Then, we have

$$\begin{aligned}\int \frac{dx}{1+\sin x+\cos x} &= \int \frac{2dt}{(1+t^2)[1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}]} = \int \frac{dt}{1+t} \\ &= \ln|1+t| = \ln|1+\tan\frac{x}{2}|\end{aligned}$$

Hence, we find

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x} = \ln|1+\tan\frac{x}{2}|_0^{\frac{\pi}{2}} = \ln 2$$

### 1.7.7 Integrating the expressions $e^{ax}\sin bx, e^{ax}\cos bx$

Let us find the indefinite integral

$$I_1 = \int e^{ax}\sin bx\ dx, \qquad I_2 = \int e^{ax}\cos bx\ dx, \quad a\neq 0, \quad b\neq 0.$$

Applying integration by pars, we note

$$u = e^{ax}, \quad u^{'} = ae^{ax}, \quad v^{'} = \sin bx, \quad v = -\frac{1}{b}\cos bx$$

Also, for the second integral

$$u = e^{ax}, \quad u^{'} = ae^{ax}, \quad v^{'} = \cos bx, \quad v = \frac{1}{b}\sin bx$$

Then, we find for both integrals

$$\int e^{ax}\sin bx\ dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx,$$
$$\int e^{ax}\cos bx\ dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx,$$

Hence, we have the system of two equations

$$I_1 = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}I_2$$
$$I_2 = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I_1$$

Solving the system of linear equations withe the unknowns $I_1$ and $I_2$, we find

$$I_1 = \int e^{ax}\sin bx\ dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C$$

$$I_2 = \int e^{ax}\cos bx\ dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C$$

**Example 1.39** *Evaluate the integrals*

$$\int_0^{\frac{\pi}{2}} e^{2x}\sin 3x\ dx, \qquad \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x\ dx.$$

**Solution**. To find the indefinite integrals, we apply integration by pars. We denote

$$u = e^{2x}, \quad u^{'} = 2e^{2x}, \quad v^{'} = \sin 3x, \quad v = -\frac{1}{3}\cos 3x$$

Also, for the second integral

$$u = e^{2x}, \quad u^{'} = 2e^{2x}, \quad v^{'} = \cos 3x, \quad v = \frac{1}{3}\sin 3x$$

Then, we find for both integrals

$$\int e^{2x}\sin 3x\ dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x,$$
$$\int e^{2x}\cos 3x\ dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x,$$

Hence, we have the system of two equations

$$I_1 = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}I_2$$
$$I_2 = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}I_1$$

Solving the system of linear equations withe the unknowns $I_1$ and $I_2$, we find

$$I_1 = \int e^{2x}\sin 3x\ dx = \frac{2\sin 3x - 3\cos 3x}{13}e^{2x} + C$$

$$I_2 = \int e^{2x}\cos 3x\ dx = \frac{2\cos 3x + 3\sin bx}{13}e^{2x} + C$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2x}\sin 3x\ dx &= \frac{2\sin 3x - 3\cos 3x}{13}e^{2x}|_0^{\frac{\pi}{2}} \\ &= \frac{-2}{13}e^{\pi} + \frac{3}{13} = \frac{1}{13}(3 - 2e^{\pi})\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{2x}\cos 3x\ dx &= \frac{3\sin 3x + 2\cos 3x}{13}e^{2x}|_0^{\frac{\pi}{2}} \\ &= \frac{-3}{13}e^{\pi} - \frac{2}{13} = -\frac{1}{13}(3e^{\pi} + 2)\end{aligned}$$

### 1.7.8 Integrals of Even and Odd Functions

.

**Even Functions**. Let $f(x)$ be a continuous even function in the symmetric interval $[-a, a]$. That is, $f(x)$ satisfies the condition

$$f(-x) = f(x) \quad \text{for all} \quad -a \leq x \leq a, \quad a > 0.$$

Then, the integral

$$\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx \tag{1.28}$$

Indeed, we note that

$$\int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_0^a f(x)\ dx \tag{1.29}$$

Let us substitute to the first integral $t = -x$, $dt = -dx$, $t = a$, when $x = -a$ and $t = 0$ when $x = 0$. Then, we find

$$\int_{-a}^{0} f(x)\ dx = -\int_a^0 f(-t)\ dt = \int_0^a f(t)\ dt$$

Hence, by (1.29), we have formula (1.28).

**Example 1.40** *Consider the function*

$$f(x) = \frac{1}{4c^2 + x^2}, \qquad -\infty < x < \infty$$

*Show that $f(x)$ is an even function on the whole real line and for all values of the parameter $c$ Evaluate the integral*

$$\int_{-2}^{2} \frac{1}{4+x^2} dx,$$

We find

$$f(-x) = \frac{1}{4c^2 + (-x)^2} = \frac{1}{4c^2 + x^2} = f(x),$$

for all $-\infty < x < \infty$.
Thus, the function is even and the integral

$$\int_{-2}^{2} \frac{1}{4+x^2} dx = 2\int_{0}^{2} \frac{1}{4+x^2} dx = \frac{2}{2} ArcTan\frac{x}{2}|_0^2 = \frac{\pi}{4},$$

**Odd Functions**. Now, let $f(x)$ be a continuous odd function in the symmetric interval $[-a, a]$. That is, $f(x)$ satisfies the condition

$$f(-x) = -f(x) \quad \text{for all} \quad -a \leq x \leq a, \quad a > 0.$$

Then, the integral

$$\int_{-a}^{a} f(x)\, dx = 0 \tag{1.30}$$

Indeed, we note that

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx \tag{1.31}$$

Let us substitute to the first integral $t = -x$, $dt = -dx$, $t = a$, when $x = -a$ and $t = 0$ when $x = 0$. Then, we find

$$\int_{-a}^{0} f(x)\, dx = -\int_{a}^{0} f(-t)\, dt = -\int_{0}^{a} f(t)\, dt$$

Hence, by (1.31), we have formula (1.30).

**Example 1.41** *Consider the function*

$$f(x) = \sin 2x \ \cos 3x, \qquad -\infty < x < \infty$$

*Show that the integral*

$$\int_{-1}^{1} \sin 2x \ \cos 3x\, dx = 0,$$

We find

$$f(-x) = \sin 2(-x)\ \cos 3(-x) = -\sin 2x\ \cos 3x = -f(x),$$

for all $-\infty < x < \infty$.
Thus, the function is odd and the integral

$$\int_{-1}^{1} \sin 2x\ \cos 3x\ dx = 0$$

## 1.8 Applications of Integrals

### 1.8.1 Area between two curves

Let us consider two continuous functions $f(x)$ and $g(x)$ in the interval $[a, b]$. Suppose that

$$f(x) \geq g(x), \qquad a \leq x \leq b.$$

Then the area between the curves is given by the formula

$$A = \int_a^b [f(x) - g(x)]dx$$

**Example 1.42** *Find the area between the parabola*

$$f(x) = 1 - x^2$$

*and the straight*

$$g(x) = x - 1$$

**Solution.** We find the points of intersection, that is, when $f(x) = g(x)$. Thus, we solve the equation

$$1 - x^2 = x - 1, \qquad \text{or} \qquad x^2 + x - 2 = (x + 2)(x - 1) = 0$$

The solution is $x = -2$ and $x = 1$. The curves intersects at the points $(-2, -3)$ and $(1, 0)$.

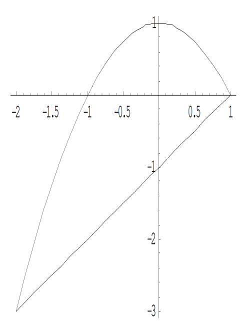



Fig. 7 Area between the curves $y = 1 - x^2$ and $y = x - 1$

By the formula, the area

$$\begin{aligned} A = \int_{-2}^{1} [(1 - x^2) - (x - 1)]\ dx &= \int_{-2}^{1} (2 - x^2 - x)\ dx \\ &= 2x - \frac{x^3}{3} - \frac{x^2}{2})|_{-2}^{1} \\ &= (2 - \frac{1}{3} - \frac{1}{2}) - (-4 + \frac{8}{3} - 2) = \frac{9}{2} \end{aligned}$$

**Example 1.43** *Find the area bounded by the graphs of the functions*

$$y = 4 - x^2, \qquad y = x^2$$

**Solution.** We find the points of intersection, that is, when

$$4 - x^2 = x^2, \qquad \text{or} \qquad 2x^2 = 4$$

The solution is $x = \sqrt{2}$ and $x = -2\sqrt{2}$. The curves intersects at the points $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$.

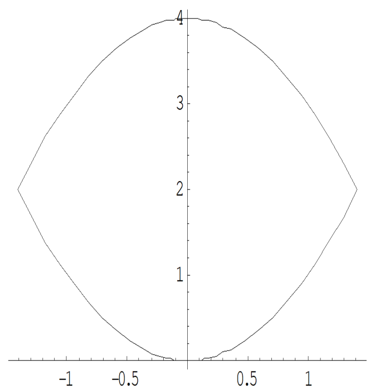



Fig. 8. Area between the curves $y = 4 - x^2$ and $y = x^2$

By the formula, the area

$$\begin{aligned} A = \int_{-\sqrt{2}}^{\sqrt{2}} [(4 - x^2) - x^2]\, dx &= \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2x^2)\, dx \\ &= (4x - \frac{2x^3}{3})|_{-\sqrt{2}}^{\sqrt{2}} \\ &= (4\sqrt{2} - \frac{4\sqrt{2}}{3}) - (-4\sqrt{2} + \frac{4\sqrt{2}}{3}) \\ &= \frac{16\sqrt{2}}{3} \end{aligned}$$

**Example 1.44** *Find the area between two parasols*

$$y^2 = 2px$$

*and*

$$x^2 = 2py$$

**Solution.** We write the equations in the form

$$y_1 = \frac{x^2}{2p} \qquad \text{and} \qquad y_2 = \sqrt{2px}$$

The curves intersect when $y_1 = y_2$. So, we solve the equation

$$\frac{x^2}{2p} = \sqrt{2px}$$

Clearly, the solutions are $x_1 = 0,\ y_1 = 0$ and $x_2 = 2p,\ y_2 = 2p$. Thus, the points of intersection are the origin $(0,0)$ and $(2p, 2p)$.

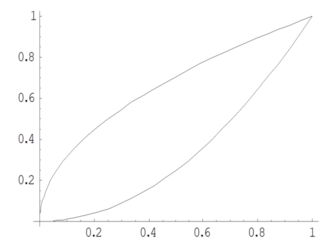



Fig. 9 Area between the curves $y = 1 - x^2$ and $y = x - 1$

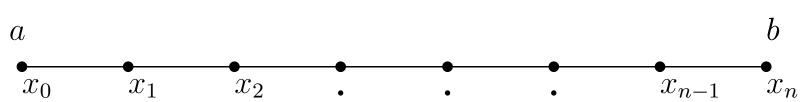
By the formula, the area

$$A = \int_0^{2p} [\sqrt{2px} - \frac{x^2}{2p}\, dx \quad = \quad [\frac{2}{3}\sqrt{2p}\, x^{\frac{2}{3}}]_0^{2p} = \frac{4}{3}p^3$$

### 1.8.2 Length of a curve

Consider a continuously differentiable function $f(x)$ in the interval $a,b]$. In order to find the length $L$ of the curve $y = f(x),\ a \leq x \leq b$, we divide the interval $[a,b]$ into subintervals, so that

$$[a,b] = [x_0, x_1] \cup [x,x_2] \cup ..., \cup [x_{n-1}, b]$$




Partition of the interval $[a,b]$

where

$$x_0 = a,\ x_{k+1} = x_k + \Delta x_k, \quad k = 0, 1, ..., n-1, \quad x_n = b,$$

Let

$$\Delta y_k = f(x_{k+1}) - f(x_k), \quad k = 0, 1, ..., n-1.$$

Using relations between sides of a right triangle, we approximate the piece $\Delta s_k$ of the curve $y = f(x)$ for $x \in [x_k, x_{k+1}]$ by the formula

$$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

So that, the length of arc $L$, we approximate by the Riemann sum

$$L \approx \sum_{k=0}^{n-1} \Delta s_k = \sum_{k=0}^{n-1} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=0}^{n-1} \sqrt{1 + (\frac{\Delta y_k)}{\Delta x_k})^2} \Delta x_k$$

of the function $\sqrt{1 + [f'(x)]^2}$.
Hence, for $\Delta x_k - > 0$, when $n - > \infty$, we obtain the formula for the length of the curve given by the graph of the function $f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \tag{1.32}$$

**Example 1.45** *Find the length of the curve given by the graph of the function*

$$f(x) = x^2$$

*which joins the points* $(-1, 1)$ *and* $(1, 1)$

**Solution**. We have $f(x) = x^2$, $f^{'}(x) = 2x$. By the formula, the length

$$L = \int_{-1}^{1} \sqrt{1 + 4x^2} dx = 2 \int_0^1 \sqrt{1 + 4x^2} dx$$

Let $u = 2x$, $du = 2dx$. Then, using formula (1.32), we find indefinite integral

$$\begin{aligned} L = \int \sqrt{1 + 4x^2} dx &= \frac{1}{2} \int \sqrt{1 + u^2} du \\ &= \frac{1}{4} [u\sqrt{1 + u^2} + ArcSinhu] + C \\ &= \frac{1}{2} x\sqrt{1 + 4x^2} + \frac{1}{4} ArcSinh2x + C \end{aligned}$$

Hence, we find the length

$$\begin{aligned} L = 2 \int_0^1 \sqrt{1 + 4x^2} dx &= [x\sqrt{1 + 4x^2} + \frac{1}{2} ArcSinh2x]_0^1 \\ &= [\sqrt{5} + \frac{1}{2} ArcSinh2]] \end{aligned}$$

### 1.8.3 Volume of a Solid.

Let $f(x) \geq 0$ be a continuous function in interval $[a, b]$. If the graph of the function $y = f(x)$, revolves about $x$-axis, then the points $(x, f(x))$ move along the circles with radius $r = f(x)$ which lies in the plane veridical to $x$-axis. Let us consider the volume of the cylinder $V_k$ bounded by the circles of radius $r = f(x)$ which lies between two planes $x = x_k$ and $x = x_{k+1}$, veridical to $x$-axis with the hight $\Delta x_k = x_{k+1} - x_k$, $k = 0, 1, ...n - 1$. Then, the volume of the cylinder is given by the formula

$$V_k = \pi f^2(x_k) \Delta x_k.$$

The approximate value of the solid $V$ is the Riemann sum

$$V \approx \sum_{k=0}^{n-1} \pi f^2(x_k)\Delta x_k,$$

of the function $\pi f^2(x)$.
In the limit, when $n- > \infty$, $\Delta x_k- > 0$, we obtain the formula for the volume of the solid generated by revolving the area below the graph of the function $f(x)$ about x-axis

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

Similarly, the volume of the solid generated by revolving about $y$-axis the area between $y-$ axis and the graph of the function $x = g(y)$, $c \leq y \leq d$, is given by the formula

$$V = \pi \int_c^d [g(y)]^2 dy.$$

Now, let us consider the area between two graphs of functions $f(x)$ and $g(x)$, $f(x) \geq g(x) \geq 0$, $a \leq x \leq b$. Then, the volume of the solid generated by the area about $x$-axis is given by the formula

$$V = \pi \int_a^b [f^2(x) - g^2(x)] dx.$$

**Example 1.46** *The area under the graph of the function*

$$f(x) = \sqrt{x}, \qquad 0 \leq x \leq 4.$$

*is rotated about the x-axis. Find the volume of the solid generated.*
**Solution**. *Straight foreword application of the formula leads us to the result*

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{x^2}{2}|_0^4 = 8\pi$$

**Example 1.47** *Find the volume of a cone which has hight $h$ and the radius $r$.*

**Solution**. Let the vertex of the cone be at the origin and its hight along the x-axis. Then, the equation of the side is

$$y = \frac{r}{h}x$$

By the formula the volume

$$V = \pi \int_0^h (\frac{r}{h})^2 \, dx = \frac{\pi r^2}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{1}{3}\pi r^2 h.$$

**Example 1.48** *Find the volume of the solid generated by revolving the graph of half of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

*about the x-axis.*

**Solution**. Half of the ellipse is given by the function

$$y = b\sqrt{1 - \frac{x^2}{a^2}}, \qquad -a \leq x \leq a, \ \ a > 0, \ b > 0.$$

By the formula

$$\begin{aligned} V = \pi b^2 \int_{-a}^{a} (\sqrt{(1 - \frac{x^2}{a^2}})^2 \, dx \ &= \ \pi b^2 \int_{-a}^{a} (1 - \frac{x^2}{a^2}) \, dx \\ &= \ \pi b^2 (x - \frac{x^3}{3a^2}|_{-a}^{a} = \frac{4}{3} ab^2 \pi \end{aligned}$$

### 1.8.4 Volume of Cylindrical Shells

Let $f(x) \geq 0$ be a continuous function in interval $[a, b]$. Now, let the graph of the function $y = f(x)$, revolves about $y$-axis, then the points $(x, f(x))$ move along the circles with radius $r = x$ which lies in the plane veridical to $y$-axis. We consider strip area bounded by two discs of radius $r_1 = x$ and $r_2 = x + \Delta x$. Then, the area of the ring bounded by the two circles of radius $r_1$ and $r_2$ is given by the formula

$$\Delta A = \pi(r_1^2 - r_2^2) = 2\pi \frac{(r_1 + r_2)}{2} (r_1 - r_2) = 2\pi r \Delta x$$

where

$$r = \frac{r_1 + r_2}{2} = x + \frac{\Delta x}{2}$$

So that, the volume of the ring is given by the formula

$$\Delta V = 2\pi xy \ \Delta x$$

The approximate value of the volume of the solid $V$ is the Riemann sum

$$V \approx \sum_{k=0}^{n-1} 2\pi x_k f(x_k) \Delta x_k.$$

of the function $2\pi x f(x)$.
In the limit, when $n- > \infty$, $\Delta y_k - > 0$, we obtain the formula for the volume of the solid generated by revolving the area below the graph of the function $f(x)$ about y-axis

$$V = 2\pi \int_a^b xf(x)\, dx \tag{1.33}$$

Similarly, revolving the area between $y$ axis and the graph of the function $x = g(y),\ c \le y \le d$ about $x-$ axis, we obtain the solid of the volume

$$V = 2\pi \int_c^d yg(y)\, dy \tag{1.34}$$

**Example 1.49** .

*(a) Evaluate the volume of the solid generated by revolving the area between $x$- axis and the graph of the function $y = 4 - x^2$ about the $y-$ axis.*

*(b) Evaluate the volume of the solid generated by revolving the area between $y$- axis and the graph of the function $x = 4 - y^2$ about the $x-$ axis.*

**Solution (a).** We find the points of intersection of the function $y = 4 - x^2$ and the $x-$ axis. Solving the equation

$$4 - x^2 = 0$$

we obtain the points of intersection $(-2, 0)$ and $(2, 0)$.
Because the function $f(x) = 4 - x^2$ is symmetric about $x-$ axis, therefore, we consider the interval of integration $[0, 2]$. Then, by the formula (1.33), we evaluate

$$\frac{1}{2}V = 2\pi \int_0^2 x(4 - x^2)\, dx = \pi[2x^2 - \frac{x^4}{4}]|_0^2 = 8\pi$$

Hence, the volume $V = 16\pi$.
**Solution (b).** Let us find the points of intersection of the function $x = 4 - y^2$ and the $y-$ axis. Solving the equation

$$4 - y^2 = 0$$

we find the points of intersection $(0, -2)$ and $(0, 2)$. Because the function $g(y) = 4 - y^2$ is symmetric about $y-$ axis, therefore, we consider the interval of integration $[0, 2]$. By the formula (1.34), we evaluate the volume

$$\frac{1}{2}V = 2\pi \int_0^2 y(4 - y^2)\, dy = \pi[2y^2 - \frac{y^4}{4}]|_0^2 = 8\pi$$

Hence, the volume $16\pi$.
**Choice of a Formula.** In order to evaluate a volume generated by revolving a region about $x-$ axis or $y-$ axis, we have the following options:

1. When the region bounded by the curve $y = f(x) \ge 0,\ a \le x \le b$, revolves about $x-$ axis, the volume

$$V = \pi \int_a^b [f(x)]^2 dx$$

2. When the region bounded by the curve $y = f(x) \geq 0,\ a \leq x \leq b$, revolves about $y-$ axis, the volume
$$V = 2\pi \int_a^b xf(x)]dx$$
3. When the region bounded by the curve $x = g(y) \geq 0,\ c \leq y \leq d$, revolves about $y-$ axis, the volume
$$V = \pi \int_c^d [g(y)]^2 dy$$
4. When the region bounded by the curve $x = g(y),\ y \geq 0,\quad c \leq y \leq d$, revolves about $x-$ axis, the volume
$$V = 2\pi \int_c^d yg(y)dy$$

### 1.8.5 Surface of a solid

Let us consider a differentiable function $f(x) \geq 0$ in the interval $[a, b]$. Then, the approximate value $S_n$ of the surface $S$ of the union of solids $V_k,\ k = 0, 1, ..., n-1$ generated by rotating about $x$-axis of the curves
$$y_k = f(x), \quad \text{for} \quad x_k \leq x \leq x_{k+1},\ k = 0, 1, ..., n-1,$$
is the Riemann sum
$$S_n = 2\pi \sum_{k=0}^{n-1} f(x_k)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}\Delta x_k, \qquad k = 0, 1, 2, ..., n-1$$
of the function $2\pi f(x)\sqrt{1 + [f'(x)]^2}$.
Hence, in the limit, when $n- > \infty$, we obtain the following formula for the surface $S$ of the solid generated by rotating about $x$-axis the area between the graph of the function $y = f(x),\ a \leq x \leq b$ and $x-$axis
$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx.$$
**Example 1.50** *Find the surface of the ball generated by rotation of the circle $x^2 + y^2 = r^2$ about $x$-axis.*

Consider the function
$$y = f(x) = \sqrt{r^2 - x^2}, \qquad 0 \leq x \leq r.$$
We have
$$f(x) = \sqrt{r^2 - x^2}, \qquad f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}$$
By the formula, half of the surface $S$ of the ball is
$$\frac{S}{2} = 2\pi \int_0^r \sqrt{r^2 - x^2}\sqrt{1 + [\frac{-x}{\sqrt{r^2 - x^2}}]^2}dx = 2\pi \int_0^r x\ dx = 2\pi r^2.$$
Hence, the surface of the ball is $S = 4\pi r^2$.

### 1.8.6 Moments and Center of Mass

Let $f(x) \geq 0$ be a continuous function in $[a,b]$. The moments $M_x$ and $M_y$ and the coordinates $(\xi,\eta)$ of the center of the mass of $A=\int_a^b f(x)dx$ under the curve $y=f(x),\ x\in[a,b]$ are defined by the formulae

$$M_x = \frac{1}{2}\int_a^b f^2(x)\,dx, \qquad M_y = \int_a^b x f(x)\,dx$$
$$\xi = \frac{M_y}{A} = \frac{1}{A}\int_a^b x f(x)\,dx, \qquad \eta = \frac{M_x}{A} = \frac{1}{2A}\int_a^b f^2(x)\,dx$$

Also, the volume $V$ of revolution of the curve $y=f(x),\ x\in[a,b]$, about $x$-axis is given by the formula $V = 2\pi\eta A$.

**Example 1.51** .

(a) *Find moments $M_x$ and $M_y$ and the coordinates of the center of mass of the area between parabola $y^2 = 2p\,x$ and $x$-axis for $x \in [0,1]$.*

(b) *Evaluate the volume of revolution of the parabola $y^2 = 2p\,x,\ x \in [0,1]$*

We have $y = f(x) = \sqrt{2px},\ \ p>0, \quad 0 \leq x \leq 1$.
By the formula

$$M_x = \frac{1}{2}\int_a^b [f(x)]^2 dx = \frac{1}{2}\int_0^1 2px\,dx = \frac{1}{2}p$$
$$M_y = \int_a^b x f(x)dx = \int_0^1 \sqrt{2p}x\sqrt{x}\,dx = \frac{2}{5}\sqrt{2p}.$$

The area of the region under the graph of the function $f(x) = \sqrt{2px}, \quad 0 \leq x \leq 1$ is

$$A = \int_0^1 \sqrt{2px}\,dx = \frac{2}{3}\sqrt{2p}$$

Then , we evaluate the coordinates $(\xi,\eta)$ of the center of mass

$$\xi = \frac{1}{A}\int_0^1 x f(x)\,dx = \frac{3}{2\sqrt{2p}}\int_0^1 x\sqrt{2px}\,dx = \frac{3}{5}$$
$$\eta = \frac{1}{2A}\int_0^1 [f(x)]^2\,dx = \frac{3}{2\sqrt{2p}}\int_0^1 2px\,dx = \frac{3}{8}\sqrt{2p}$$

By the formula, volume of the solid generated by revolving the area about $x$-axis

$$V = 2\pi\eta\,A = 2\pi\,\frac{3}{8}\sqrt{2p}\,\frac{2}{3}\sqrt{2p} = \pi\,p$$

**Center of Mass of a Region Bounded by Two Curves**. Let us consider two continuous functions $f(x) \geq g(x)$ in the interval $[a,b]$. The coordinates $\xi,\eta$ of the center of mass (centroid) of the region between the graphs of the two functions is given by the formulae:

$$\xi = \frac{M_y}{A}, \qquad \eta = \frac{M_x}{A}$$

where the area of the region

$$A = \int_a^b [f(x) - g(x)]dx$$

and the moments of the region

$$M_y = \int_a^b x[f(x) - g(x)]dx,$$
$$M_x = \frac{1}{2}\int_a^b [f(x) + g(x)][f(x) - g(x)]dx = \frac{1}{2}\int_a^b [f^2(x) - g^2(x)]dx$$

**Example 1.52** *Find the center of mass (centroid) of the region bounded by the curves $y = x^3$ and $y = \sqrt{x}$*

**Solution.** We find the points of intersection of the curves $f(x) = x^3$ and $g(x) = \sqrt{x}, x \geq 0$. Solving the equation $x^3 = \sqrt{x}$, we find the solution $x = 0$ or $x = 1$. Thus, the curves intersect at the points $(0, 0)$ and $(1, 1)$. We note that $\sqrt{x} \geq x^3$ for $0 \leq x \leq 1$. First, we evaluate the area of the region

$$A = \int_0^1 [\sqrt{x} - x^3]dx = \frac{5}{12}$$

Then, we find the moments of the centroid

$$M_y = \int_0^1 x(\sqrt{x} - x^3)dx = \int_0^1 x^{\frac{3}{2}} - x^4)dx = \frac{1}{5}$$
$$M_x = \frac{1}{2}\int_0^1 (x - x^6)dx = \frac{5}{28}$$

Hence, we obtain the coordinates of the center of mass

$$\xi = \frac{M_y}{A} = \frac{12}{25}, \qquad \eta = \frac{M_x}{A} = \frac{3}{7}.$$

## 1.9 Numerical Integration

### 1.9.1 Trapezoidal Rule.

Let $f(x)$ be a given function twice contiguously differentiable in the interval$[a, b]$. We consider uniform partition of the interval $[a, b]$ by the points

$$x_i = a + i\,h, \quad h = \frac{b-a}{n}, \quad i = 0, 1, 2, ..., n.$$

Then, the area of the tropez with the base $h = x_{i+1} - x_i$ and the parallel sides $f(x_{i-1})$ and $f(x_i)$ is given by the formula

$$T_i = \frac{f(x_{i-1}) + f(x_i)}{2}h, \qquad i = 1, 2..., n$$

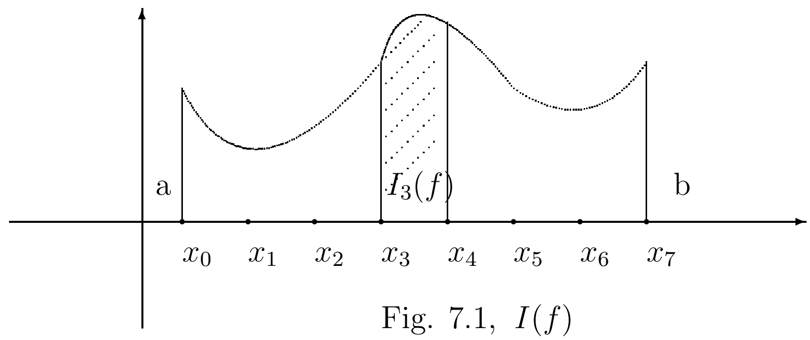



Fig. 7.1, $I(f)$

The approximated area under the graph of the function is the sum of the areas of $n$ trapeze, that is

$$T = T_1 + T_2 + \cdots + T_n$$

So that

$$\begin{aligned} T &= \frac{f(x_0)+f(x_1)}{2}h + \frac{f(x_1)+f(x_2)}{2}h \\ &+ \frac{f(x_2)+f(x_3)}{2}h + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}h, \\ &= \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)] \end{aligned} \tag{1.35}$$

In this way, we arrived at the composed trapezoidal rule

$$T_h(f) = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

where the truncation error

$$E_T(f,h) = -\frac{h^2}{12}(b-a)f''(\eta),$$

for certain $\eta \in (a,b)$, so that

$$I(f) = T_h(f) + E_T(f,h).$$

The truncation error of the trapezoidal rule satisfies the following inequality:

$$\mid E_T(f,h) \mid \leq \frac{M^{(2)}}{12}(b-a)h^2,$$

where

$$M^{(2)} = \sup_{a\leq x\leq b} \mid f''(x) \mid .$$

**Example 1.53** *Evaluate the integral*

$$\int_0^2 ln(1+x)dx$$

*by trapezoidal rule with the accuracy* $\epsilon = 0.05$.

**Solution.** In order to get accuracy $\epsilon = 0.05$, we shall estimate the step-size $h$, so that, we choose the greatest $h = \frac{b-a}{n}$ for which the following inequality holds:

$$E_T(f,h) \leq \frac{h^2}{12}(b-a)M^{(2)} \leq \epsilon.$$

Because

$$f(x) = ln(1+x), \quad f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2},$$

we have

$$M^{(2)} = \max_{0\leq x\leq 2} \frac{1}{(1+x)^2} = 1.$$

So, the inequality

$$E_T(f,h) \leq \frac{h^2}{12}2 < 0.05$$

holds for $h = 0.5$ and $n = 4$.
The approximate value of the integral is:

$$\begin{aligned} T(f) &= 0.25[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \\ &= 0.25[ln(1) + 2ln(1.5) + 2ln(2) + 2ln(2.5) + ln(3)] = 1.282105. \end{aligned}$$

### 1.9.2 Simpson Rule.

Let $y = f(x)$ be a four times continuously differentiable function in the interval$[a,b]$. We consider uniform partition of the interval $[a,b]$, by $2n+1$ points

$$a = x_0 < x_1 < x_2 <, ..., < x_{2n-1} < x_{2n} = b$$

So that

$$x_i = a + ih, \ i = 0,1,2,...,2n, \quad h = \frac{b-a}{2n}$$

and the interval consists of $2n$ subintervals

$$[a,b] = [a,x_1] \cup [x_1,x_2] \cup [x_2,x_3] \cup \cdots \cup [x_{2n-1},x_{2n}].$$

In order to derive Simpson rule, we consider the area $S_1$ between x-axis and the graph of the quadratic function

$$\begin{aligned} y(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \\ &= \frac{(x-x_1)(x-x_2)}{2h^2}f(x_0) - \frac{(x-x_0)(x-x_2)}{h^2}f(x_1) + \frac{(x-x_0)(x-x_1)}{2h^2}f(x_2) \end{aligned}$$

Let us note that the quadratic function $y(x)$ goes thought the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

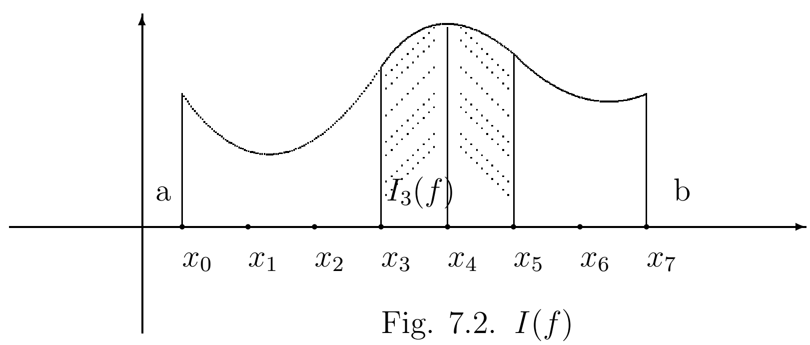



Fig. 7.2. $I(f)$

The area is given by the integral

$$\begin{aligned} S_1 = \int_{x_0}^{x_2} f(x)\,dx &= \frac{f(x_0)}{2h^2}\int_{x_0}^{x_2}(x-x_1)(x-x_2)\,dx \\ &- \frac{f(x_1)}{h^2}\int_{x_0}^{x_2}(x-x_0)(x-x_2)\,dx \\ &+ \frac{f(x_2)}{2h^2}\int_{x_0}^{x_2}(x-x_0)(x-x_1)\,dx \\ &= \frac{f(x_0)}{2h^2}\frac{2h^3}{3} + \frac{f(x_1)}{h^2}\frac{4h^3}{3} + \frac{f(x_2)}{2h^2}\frac{2h^3}{3} \\ &= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \end{aligned}$$

Then, the area between x-axis and graph of the function $f(x)$ over the subinterval $[x_0, x_2]$ is the integral

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

This is the simple Simpson's method. The error of the simple Simpson's method

$$\int_{x_0}^{x_2} f(x)dx - \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] = -\frac{h^5}{90}f^{(4)}(\eta)$$

To obtain the composed Simpson's method, we apply the simple Simpson's method to each of the integrals on the right side

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx,$$

So that

$$\int_{x_{2i-2}}^{x_{2i}} f(x)\,dx \approx \frac{h}{3}(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i+1})) = S_i$$

for $i = 1, 2, ..., n$.
Then, we obtain composed Simpson rule

$$S_h(f) \quad = \quad S_1 + S_2 + \cdots + S_n,$$

that is

$$\begin{aligned} S_h(f) &= \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) \\ &+ 2f(x_4) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)] \end{aligned}$$

Hence, the integral

$$\int_a^b f(x)dx = S_h(f) + E_S(f, h),$$

where the error

$$E_S(f, h) = -\frac{h^5}{90}[f^{(4)}(\eta_1) + f^{(4)}(\eta_2) + \cdots + f^{(4)}(\eta_n)],$$

for $\eta_i \in [x_{2i-2}, x_{2i}],\ i = 1, 2, ..., n$
Hence, by the intermediate value theorem, there exists $\eta \in (a, b)$ such that

$$f^{(4)}(\eta_1) + f^{(4)}(\eta_2) + \cdots + f^{(4)}(\eta_n) = nf^{(4)}(\eta) = \frac{b-a}{2h}f^{(4)}(\eta),$$

Therefore, the error of the composed Simpson rule is:

$$E_S(f, h) = -\frac{h^5}{90}nf^{(4)}(\eta) = -\frac{h^4}{180}(b-a)f^{(4)}(\eta),$$

for certain $\eta \in (a, b)$.
This error satisfies the following inequality:

$$\mid E_S(f, h) \mid \leq \frac{h^4}{180}(b-a)M^{(4)}. \tag{1.36}$$

where

$$M^{(4)} = \max_{a \leq x \leq b} \mid f^{(4)}(x) \mid .$$

**Example 1.54** *Evaluate the integral*

$$I(f) = \int_0^2 ln(1+x)dx$$

*by Simpson rule using step-size* $h = 0.5$. *Estimate the truncation error* $E_S(f,h)$.

**Solution.** We note that $2n = \dfrac{b-a}{h} = 4$ and

$$\begin{aligned} S_h(f) &= \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{0.5}{3}[ln(1) + 4ln(1.5) + 2ln(2) + 4ln(2.5) + ln(3)] = 1.295322 \end{aligned}$$

The exact value of $I(f) = 1.295837$, so that the error $E_S(f,h) = I(f) - S(f) = 1.295837 - 1.295322 = 0.000515$.
Also, we note that

$$f^{(4)}(x) = -\frac{6}{(1+x)^4} \quad \text{and} \quad M^{(4)} = \max_{0\leq x\leq 2} \frac{6}{1+x)^4} = 6.$$

Hence

$$\mid E_S(f,h) \leq \frac{h^4}{180}(b-a)M^{(4)} = \frac{0.0625}{180} 2 * \ 6 = 0.00417.$$

**Question 1.1** .

(a) *Assume that the error* $E_T(f,h)$ *of the trapezoidal method* $T(f,h)$ *is proportional to* $h^2$, *so that*

$$E_T(f,h) = C\ h^2$$

*for a constant* $C$.
*Show that the*

$$E_T(f,2h) = 4\ E_T(f,h)$$

(b) *Let* $2n$ *be an even number of the subinterval in an uniform portion of the interval* $[a,b]$. *Denote by* $T(f,2h)$, $T(f,h)$ *two trapezoidal results obtained for* $2h$ *and* $h = \dfrac{b-a}{2n}$, *respectively. Show that the Simpson method*

$$S(f,h) = \frac{1}{3}(4T(f,h) - T(f,2h))$$

**Solution**.
**To(a).** By the assumption

$$E_T(f,h) = C\ h^2, \qquad E_T(f,2h) = 4C\ h^2$$

Hence, we find

$$E_T(f,2h) = 4C\ h^2 = 4E_T(f,h)$$

**To(b)**. The formulae of the trapezoidal method defined on $2n$ subintervals are:

$$\begin{aligned} T(f,h) &= \frac{h}{2}[f(a)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{2n-1})+f(b)] \\ T(f,2h) &= h[f(a)+2f(x_2)+2f(x_2)+2f(x_4)+\cdots+2f(x_{2n-2})+f(b)] \end{aligned}$$

Hence, we find

$$\begin{aligned} \frac{1}{3}[4T(f,h)-T(f,2h)] &= \frac{h}{2}[f(a)+2f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)\cdots \\ &+ 4f(x_{2n-2})+2f(x_{2n-1})+f(b)] = S(f,h) \end{aligned}$$

## 1.10 Exercises. Set 1

### Riemann Sums and Riemann Integral.

**Question 1.2** .

(a) *State the definition of Riemann sums of a function $f(x)$ given in the interval $[a,b]$.*

(a) *Find the Riemann sums of the following functions on an uniform partition of the interval $[a,b]$*

$$(i) \quad f(x) = x^2+x+1, \quad 0 \le x \le 1, \qquad (ii) \quad g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ x, & 1 < x \le 2 \end{cases}$$

(c) *Find the limit of the Riemann sums of the functions $f(x)$ and $g(x)$.*

**Question 1.3** .
*Use the Riemann sums*

(a) *to find the area between $x$-exis and the graph of the function*

$$f(x) = 1+x^2, \quad 0 \le x \le 1.$$

(b) *Evaluate the mean value of the function*

$$f(x) = 2x^2, \qquad 0 \le x \le 4,$$

*where the mean value of $f(x)$ in $[a,b]$ is given by the formula*

$$(mean(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

**Question 1.4** *Evaluate the integrals*

(a)

$$\int_0^{\frac{\pi}{2}} \cos x \; dx$$

(a)

$$\int_0^{\frac{\pi}{2}} \sin x \; dx$$

*Using the Riemann sums with the mid-points* $x_i^* = \dfrac{x_{i+1} + x_i}{2}$, *of the subintervals* $[x_i, x_{i+1}],\; i = 0, 1, ..., n-1$ *and the formulae*

$$\cos x + \cos 3x + \cos 5x + ... + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x}$$

$$\sin x + \sin 3x + \sin 5x + ... + \sin(2n-1)x = \frac{1 - \cos nx}{2 \sin x}$$

$$\lim_{x->0} \frac{\sin x}{x} = 1$$

**Fundemental Theorem of Calculs.**

**Question 1.5** .

(a) *State the Fundamental theorem of Calculus.*

(b) *Use the theorem to evaluate the following integrals*

$$\int_0^1 (1 + \sqrt{x} + \sin \pi x) \; dx$$

(c)

$$\int_1^2 \frac{x \; dx}{1+x}$$

(d)

$$\int_0^2 \frac{dx}{4+x^2}$$

(e)

$$\int_1^2 \frac{dx}{\sqrt{9-x^2}}$$

(f)

$$\int_1^3 (x^\alpha + x^\beta) \; dx, \qquad \alpha = \frac{1}{2}, \quad \beta = \frac{2}{3}$$

(g)

$$\int_0^1 (\sin \frac{\pi x}{4} + \cos \frac{\pi x}{2}) \; dx$$

*(h)*

$$(i)\quad \int_0^{\frac{\pi}{4}} \sec^2 x\, dx, \qquad (ii)\quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x\, dx$$

**Question 1.6** .
*Find the area bounded by the given curves and the given lines. Sketch the graph*

*(a)*

$$f(x) = 9 - x^2 \quad \text{and} \quad x - axis$$

*(b)*

$$f(x) = -x^2 + 6x - 5 \quad \text{and} \quad x - axis$$

*(c) Find the area between two functions*

$$f(x) = 1 - x \quad \text{and} \quad g(x) = 1 - x^2$$

**Method by Substitution.**

**Question 1.7** .
*Use the substitution to evaluate the integrals*

*(a)*

$$\int_0^4 \sqrt{1+2x}\, dx, \quad t = 1 + 2x$$

*(b)*

$$\int_0^a \frac{\sqrt{x}}{1+x}\, dx, \quad t = \sqrt{x}, \quad a > 0$$

*(c)*

$$\int_0^2 x\sqrt{4-x^2}\, dx, \quad t = \ 4 - x^2$$

*(d)*

$$\int_0^2 \frac{x^2}{4+x^2}\, dx, \quad t = \ 4 + x^2$$

*(f)*

$$\int_0^a \sqrt{a^2 - x^2}\, dx, \quad x = a\cos t, \quad a > 0$$

**Question 1.8** .
*Use the substitution $u = g(x)$ to evaluate the integrals*

*(a)*

$$\int_0^1 (2x+1)\sqrt{x^2+x+3}\, dx$$

*(b)*

$$\int_1^2 \frac{t+3t^2}{\sqrt{t^2+2t^3}}\,dt$$

*(c)*

$$\int_0^1 \sqrt{4-x^2}\,dx$$

*(d) Show that*

$$\int_0^1 \sin \pi x \cos^n \pi x\,dx = \frac{1}{\pi(n+1)}[1+(-1)^n], \qquad n = 1, 2, ...,$$

*(e) Use the substitution* $u = \tan \frac{x}{2}$ *to evaluate the integrals*

(i) $\int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{1+\sin x}\,dx$, (ii) $\int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{1+\cos x}\,dx$

(iii) $\int_0^{\frac{\pi}{4}} \frac{dx}{1+\sin x+\cos x}\,dx$, (iv) $\int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{1-\sin^2 x+\cos^2 x}\,dx$

**Question 1.9** .

*(a) For a given continuous function* $f(x)$ *in the interval* $[a, b]$, *show that*

$$\int_a^b 3x^2 f(x^3)\,dx = \int_{a^3}^{b^3} f(u)\,du$$

*(b) Let* $f(x) = e^x$. *Evaluate the integral*

$$\int_0^1 3x^2 f(x^3)\,dx$$

**Metod of Integration by Parts.**

**Question 1.10** .

*Evaluate the following integrals:*

*(a)*

$$\int_0^1 \sin \pi x \ \cos \pi x\,dx$$

*(b)*

$$\int_1^2 x \cos \frac{\pi x}{2}\,dx$$

*(c)*

$$\int_0^1 \cos^2 \pi x\,dx$$

(d)

$$(i)\quad \int_0^{\pi} e^{2x}\sin 4x\ dx,\qquad (ii)\quad \int_0^{\pi} e^{3x}\cos 6x\ dx$$

(e)

$$(i)\quad \int_0^{1} arectan\ 2x\ dx,\qquad (ii)\quad \int_0^{1} x^2\ln(1+x)\ dx$$

*(f) Show that*

$$(i)\qquad \int_{-1}^{1}\sin^2 2\pi x\ dx = \int_{-1}^{1}\cos^2 2\pi x\ dx = 1$$

(g)

$$(ii)\qquad \int \cos^3 x\ dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin\ x + C$$

*where $C$ is a constant.*

*(b) Evaluate the integral*

$$\int_0^{\frac{\pi}{2}}\cos^3 x\ dx$$

**Integration of Rational Functions**

**Question 1.11** .
*Evaluate the following integrals*

(a)

$$\int_0^1 \frac{x}{x^2-7x+10}dx$$

(b)

$$\int_{-1}^1 \frac{2x+1}{x^2+6x+9}$$

(c)

$$\int_{-1}^1 \frac{dx}{x^2-2x+5}$$

(d)

$$\int_0^1 \frac{4x\ dx}{x^2+4}$$

(e)

$$\int_0^1 \frac{4x\ dx}{(x^2+4)^2}$$

(f)

$$\int_1^2 \frac{dx}{(x^2-9)(x+2)}$$

**Trapezoidal and Simpsons Methods**

**Question 1.12** .

*(a) State the formula for $T(f,h)$ of the trapezoidal method and give an estimate of its error $error_T(f,h)$.*

*(b) Evaluate the approximate value of the integrals, when $n = 4$, by the trapezoidal method*

$$(i)\quad \int_0^{\pi} \sin x\, dx \qquad (ii)\quad \int_0^1 e^{x^2} dx \qquad (iii)\quad \int_0^1 \frac{dx}{\sqrt{1+x^3}}$$

*(c) Evaluate the integral*

$$\int_0^1 (1 + x + x^2 + x^3 + x^4)\, dx$$

*by trapezoidal method with accuracy $\epsilon = 0.08$.*

$$Answer = 2.31328,\ n = 5,\ h = 0.2$$

**Question 1.13** .

*(a) State the formula for $S(f,h)$ of Simpsson method and give an estimate of its error $error_S(f,h)$.*

*(b) Evaluate the approximate value of the integrals, when $n = 4$, by Simpson's method*

$$(i)\quad \int_0^{\pi} \sin x\, dx \qquad (ii)\quad \int_0^1 e^{x^2} dx \qquad (iii)\quad \int_0^1 \frac{dx}{\sqrt{1+x^3}}$$

*(c) Evaluate the integral*

$$\int_0^1 (1 + x + x^2 + x^3 + x^4)\, dx$$

*by Simpson method with accuracy $\epsilon = 0.05$*

$$Answer = 2.29167,\ n = 1,\ h = 0.5$$

**Question 1.14** .

*(a) Assume that the error $E_T(f,h)$ of the trapezoidal method $T(f,h)$ has the series expansion*

$$E_T(f,h) = c_2\, h^2 + c_4\, h^4 + c_6\, h^6 + \cdots$$

*Let $2n$ be an even number of the subinterval in an uniform partion of the interval $[a,b]$. Denote by $T(f,2h)$, $T(f,h)$ two trapezoidal results obtained for $2h$ and $h = \dfrac{b-a}{2n}$, respectively. Show that the error of method*

$$S(f,h) = \frac{1}{3}(4T(f,h) - T(f,2h))$$

*is*

$$E(f,h) \approx -4c_4\ h^4$$

**Question 1.15** *Sketch the graph of the curves and evaluate the area between them*

(a)

$$f(x) = x^2, \qquad g(x) = 2 - x^2$$

(b)

$$f(x) = \frac{1}{1+x^2}, \qquad g(x) = \frac{1}{2}x^2$$

,

# Chapter 2

# Ordinary Differential Equations (ODE)

## 2.1 Introduction to First Order ODE

The general form of the first order differential equations

$$\frac{dy(x)}{dx} = f(x, y(x)), \qquad a \leq x \leq b. \tag{2.1}$$

where the function $f(x, y)$ of two variable $x$ and $y$ is given in the rectangle

$$\Omega = \{(x, y) : \ a \leq x \leq b, \ \ c \leq y \leq d$$

and $y(x)$ is the unknown function of the variable $x$.
We shall see that there are many solutions of the equation. In fact, there is one parameter family of solutions of the equation. We can choose one from the family by the initial condition

$$y(a) = y_0 \quad \text{ for a given value } y_0$$

The following Cauchy-Picard theorem holds:

**Theorem 2.1** *If the function $f(x, y)$ is continuous in the c rectangle $\Omega$ with respect to both variables $x$ and $y$, and satisfies Lipschitz's condition with respect to the variable $y$, that is*

$$|f(x, y_1)f(x, y_2)| \leq L\ |y_1 - y_2| \quad \text{for all} \quad a \leq x \leq b,$$

*here the Lipschitz's constant $L > 0$ is independent of $x$ and $y$,*
*then there exists a unique solution $y(x)$ of equation*

$$\frac{dy(x)}{dx} = f(x, y(x)),$$

*which satisfies the initial condition*

$$y(a) = y_0 \tag{2.2}$$

## 2.2 First Order Linear Equation

In general, equations of the form (2.1) are successfully solvable for some types of the given function $f(x,y)$. For example, when the function $f(x,y)$ is linear in the variable $y$, that is

$$f(x,y) = -p(x)y + q(x), \qquad a \le x \le b,$$

where $p(x)$ and $q(x)$ are given in $[a,b]$.
Then, the linear equation takes the form

$$\frac{dy(x)}{dx} + p(x)y = q(x), \qquad a \le x \le b. \tag{2.3}$$

Below, we shall solve the linear differential equation (2.3). First, let us solve the homogeneous linear equation

$$\frac{dy(x)}{dx} + p(x)y = 0, \qquad a \le x \le b. \tag{2.4}$$

when the function $q(x) = 0$ for all $x$.
So, we write the equation in terms of the differentials

$$\frac{dy}{y} = -p(x)dx, \qquad a \le x \le b. \tag{2.5}$$

Integrating both sides, we find

$$\ln|y| = -\int p(x)\, dx + C_0$$

or

$$y(x) = e^{-\int p(x)\, dx + C_0} = Ce^{-\int p(x)\, dx}$$

for constant $C = e^{C_0}$.

**Example 2.1** *Find all solutions of the equation*

$$\frac{dy}{dx} - 2xy = 0, \qquad -\infty < x < \infty.$$

We write the equation in the differentials form

$$\frac{dy}{y} = 2x\, dx, \qquad -\infty < x < \infty.$$

Integrating both sides, we find

$$\ln|y| = x^2 + C_0$$

Hence, we obtain all solutions

$$y(x) = C\, e^{x^2 + C_0} = C\, e^{x^2}$$

for constant $C = e^{C_0}$.
We note that all solutions are given in the form of one parameter family of solutions with the parameter $C$.
Now, let us solve the non-homogeneous equation. Multiply equation (2.3) by the factor $e^{\int p(x)\,dx}$, we obtain

$$e^{\int p(x)dx}\frac{dy(x)}{dx} + p(x)e^{\int p(x)dx}y = e^{\int p(x)dx}q(x), \quad a \le x \le b.$$

The left side of the equation is the derivative of the product

$$\frac{d}{dx}[e^{\int p(x)dx}y] = e^{\int p(x)dx}q(x).$$

Integrating both sides, we obtain

$$e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx + C.$$

Hence, the solution

$$y(x) = e^{-\int p(x)dx}[\int e^{\int p(x)dx}q(x)dx + C],$$

or

$$y(x) = e^{-\int p(x)dx}\int e^{\int p(x)dx}q(x)dx + C\ e^{-\int p(x)dx}, \quad a \le x \le b.$$

for a constant $C$.

**Example 2.2** *Find all solutions of the equation*

$$\frac{dy}{dx} + 2y = x, \qquad -\infty < x < \infty$$

We have $p(x) = 2,\ q(x) = x$. Let us multiply the equation by the factor $e^{-\int p(x)dx} = e^{-\int 2dx} = e^{2x}$, to get

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = x\ e^{2x}, \qquad -\infty < x < \infty$$

Now, the left side of the equation is the derivative of the product

$$\frac{d}{dx}[e^{2x}y] = x\ e^{2x}.$$

Integrating both sides, we find

$$e^{2x}y = \int x\ e^{2x}dx + C$$

Hence, we have the solution

$$y(x) = e^{-2x}\int x\ e^{2x}dx + Ce^{-2x}$$

or

$$y(x) = \frac{1}{4}(2x-1) + Ce^{-2x}, \qquad -\infty < x < \infty$$

for any value of the parameter $C$.

**Example 2.3** *Find all solutions of the equation*

$$\frac{dy}{dx} + 4xy = 8x, \qquad -\infty < x < \infty$$

We have $p(x) = 4x$, $q(x) = 8x$. Let us multiply the equation by the factor $e^{\int 4xdx} = e^{2x^2}$, to get

$$e^{2x^2}\frac{dy}{dx} + 4e^{2x^2}x\, y = 8e^{2x^2}x.$$

Now, the left side of the equation is the derivative of the product

$$\frac{d}{dx}[e^{2x^2}y] = 8x\, e^{2x^2}.$$

Integrating both sides, we find

$$e^{2x^2}y = 8\int x\, e^{2x^2}dx + C$$

Hence, we have the solution

$$y(x) = 2 + Ce^{-2x^2}, \qquad -\infty < x < \infty$$

for any value of the parameter $C$.
From the examples, we observe that all solutions of a first order differential equation are in the one parameter family of solutions with the parameter $C$. However, we can choose a unique one by the initial condition, as we present in the following example

**Example 2.4** *Find the solution* $y(x)$ *of the initial value problem*

$$y' - y = \sin 2x, \qquad -\infty < x < \infty$$

$$y(0 = 1$$

We have $p(x) = -1$, $q(x) = \sin 2x$. Let us multiply the equation by the factor $e^{-\int dx} = e^{-x}$, to get

$$e^{-x}y' - e^{-x}y = e^{-x}\sin 2x.$$

Now, the left side of the equation is the derivative of the product

$$[e^{-x}y]' = e^{-x}\sin 2x.$$

Integrating both sides, we find

$$e^{-x}y = \int e^{-x}\sin x\, dx + C \tag{2.6}$$

We find the indefinite integral integrating by parts. Let

$$u = e^{-x},\ u' = -e^{-x},\ v' = \sin 2x,\ v = -\frac{1}{2}\cos 2x$$

Using the formula of integration by parts

$$\int u\, v'\, dx = u\, v - \int u'\, v\, dx$$

we find

$$I = \int e^{-x} \sin 2x\, dx = -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \int e^{-x} \cos 2x\, dx$$

Similarly, we find

$$\int e^{-x} \cos 2x\, dx = \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \int e^{-x} \sin 2x\, dx$$

Hence, we have the solution

$$\frac{5}{4} I = -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x$$

or

$$\int e^{-x} \sin 2x\, dx = -\frac{2}{5} e^{-x} \cos 2x - \frac{1}{5} e^{-x} \sin 2x. \qquad (2.7)$$

Combining (2.6) and (2.7), we find the solution

$$y(x) = -\frac{1}{5}(2\cos 2x + \sin 2x) + Ce^{x}$$

Using the initial value condition $y(0) = 1$, we determine the constant

$$y(0) = -\frac{2}{5} + C = 1, \qquad C = \frac{7}{5}$$

Hence, the unique solution

$$y(x) = -\frac{1}{5}(2\cos 2x + \sin 2x) + \frac{7}{5} e^{x}$$

satisfies the initial condition.

**Exercises**

**Question 2.1** .

(a) *Find all solutions of the equation.*

$$y'(x) + 3x^2 y(x) = x^2.$$

(b) *Find the solution $y(x)$ of the equation which satisfies the initial value condition $y(0) = 2$.*

**Question 2.2** *Find a continuous solution of the initial value problem*

$$y'(x) + y(x) = f(x), \qquad y(0) = 2$$

*where*

$$f(x) = \begin{cases} 2, & 0 \le x < 1, \\ 0, & x \ge 1 \end{cases}$$

**Question 2.3** *Show that if $w(x)$ and $v(x)$ are two solutions of the equation*

$$y'(x) + p(x)y(x) = 0$$

*then, for any constants $c_1$ and $c_2$ the function*

$$y(x) = c_1 w(x) + c_2 v(x)$$

*is also the solution of the equation*

**Question 2.4** *Prove that if $w(x)$ and $v(x)$ are two solutions of the equation*

$$y'(x) + p(x)y(x) = q(x)$$

*then the difference*

$$y(x) = w(x) - v(x)$$

*is the solution of the equation homogeneous equation*

$$y'(x) + p(x)y(x) = 0$$

## 2.3 Separable Equations

Let the function $f(x, y) = h(x)g(y)$ be the product of two function $h(x)$ and $g(y)$. So that, each of them is a function of one variable $x$ or $y$. Then, the separable equation is:

$$\frac{dy}{dx} = h(x)g(y) \tag{2.8}$$

or in terms of differentials

$$M(x)N(y)dx + m(x)n(y)dy = 0 \tag{2.9}$$

Clearly, equation(2.9) can be written in the form (2.8) that is

$$\frac{dy}{dx} = -\frac{M(x)N(y)}{m(x)n(y)}, \quad \text{for} \quad h(x) = -\frac{M(x}{m(x)}, \quad g(y) = \frac{N(y)}{n(y)} \tag{2.10}$$

provided that $m(x)n(y) \neq 0$
For example, the equation

$$\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$$

or

$$(1+x^2)dx - (1+y^2)dy = 0$$

is separable, since the function

$$f(x, y) = \frac{1+x^2}{1+y^2}$$

is the product of two functions $h(x) = 1 + x^2$ and $g(y) = \dfrac{1}{1+y^2}$ and each of them is one variable function.
In order to solve a separable equation, we write equation (2.8) in the differentials from

$$\frac{dy}{g(y)} = h(x)\, dx$$

Integrating both sides, left with respect to $y$, and right with respect to $x$, we find the solution $y(x)$ in the implicit form

$$\int \frac{dy}{g(y)} = \int h(x)\, dx$$

In the example, we find the implicit form of the solution

$$\int (1+y^2)dy = (\int 1 + x^2)\, dx + C$$

So that

$$y + \frac{y^3}{3} = x + \frac{x^3}{3} + C$$

or

$$(y - x) + \frac{1}{3}(y^3 - x^3) = C$$

for any constant $C$.
**Equations with homogeneous** $f(x, y)$. Consider the equation

$$\frac{dy}{dx} = f(x, y)$$

where the function $f(x, y)$ is homogeneous of degree $n$ if satisfies the condition

$$f(t\,x, t\,y) = t^n f(x, y)$$

for any real $t$.

**Example 2.5** *The equation*

$$\frac{dy}{dx} = \frac{y}{x+y}, \qquad x + y \neq 0,$$

is with homogeneous right side of degree 0.
We have

$$f(x, y) = \frac{y}{x+y}$$

Then

$$f(t\,x, t\,y) = \frac{(t\,y)}{t\,x + t\,y} = \frac{ty}{t\,x + t\,y} = f(x, y)$$

for any real $t$.
**General approach.** By the substitution

$$y(x) = x\ v(x)$$

we replace the unknown function $y(x)$ by the unknown function $v(x)$. Then, we have

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

and for $t = \frac{1}{x}, \quad x \neq 0$

$$f(\frac{1}{x}x, \frac{1}{x}\ y) = f(1, \frac{y}{x}) = f(1, v) = g(v)$$

Hence, we find

$$v + \frac{dv}{dx} = g(y)$$

Thus, we have arrived to the separable equation, which in terms of differentials takes the following form:

$$\frac{dv}{g(y) - v} = \frac{dx}{x}$$

Integrating both sides, we obtain the relation between $x$ and $y$ as implicit form of the solution

$$\int \frac{dv}{g(v) - v} = \ln|C\ x|$$

for any constant $C$.

**Question 2.5** *Find all solutions of the equation*

$$\frac{dy}{dx} = \frac{y}{x + y}, \qquad x + y \neq 0,$$

**Solution**. By the substitution $y = x\ v$, we transform the equation to the separable equation in $v$

$$v + x\frac{dv}{dx} = \frac{1}{1 + \frac{1}{v}}$$

or

$$\frac{1 + v}{v^2}dv = -\frac{dx}{x}, \qquad x \neq 0$$

Integrating both sides, we find the implicit form of all solutions

$$-\frac{1}{v} + \ln|v| = -\ln|C\ x|$$

$$v \ln|C\ x\ v| = 1$$

In the original variables

$$y \ln|C\ y| = x.$$

for any constant $C$.

## 2.4 Linear Ordinary Differential Equations

We shall consider the linear ordinary differential equations of the second order with constant coefficients

$$L[y](x) =\equiv \frac{d^2y}{dx^2} + p\frac{dy}{dx} + q\,y = f(x), \qquad a \le x \le b, \tag{2.11}$$

or

$$L[y](x) =\equiv y'' + p\,y' + q\,y = f(x), \qquad a \le x \le b, \tag{2.12}$$

where the constant coefficients $p,\ q$ and the right side function $f(x)$ are given, and $y(x)$ is the unknown function.
The equation is linear because the operator $L[y](x)$ in the left side is linear one. So that, the image of a linear combination

$$c_1y_1(x) + c_2y_2(x)$$

is linear combination of images, that is

$$L[c_1y_1 + c_2y_2](x) = c_1L[y_1](x) + c_2L[y_2(x)$$

Indeed, we have

$$\begin{aligned} L[c_1y_1 + c_2y_2](x) &= \frac{(d^2c_1y_1 + c_2y_2)}{dx^2} + p\frac{d(c_1y_1 + c_2y_2)}{dx} + q(c_1y_1 + c_2y_2) \\ &= c_1(\frac{d^2y_1}{dx^2} + pc_1\frac{dy_1}{dx} + qy_1) + c_2(\frac{d^2y_2}{dx^2} + c_2\frac{dy_2}{dx} + qy_2) \\ &= c_1L[y_1](x) + c_2L[y_2](x) \end{aligned}$$

for any constants $c_1$ and $c_2$.
In order to solve the equation, we begin with solution of the homogeneous equation, when the right side function $f(x) = 0$ for all $x \in [a, b]$.

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + q\,y = 0, \qquad a \le x \le b, \tag{2.13}$$

To find all solutions of the homogeneous equation, we apply Euler's substitution

$$y(x) = e^{\lambda x}$$

$$y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

Then, we obtain the equation for $\lambda$

$$y'' + py' + qy = (\lambda^2 + p\lambda + q)e^{\lambda x} = 0$$

A root $\lambda_1$ of the characteristic equation

$$\lambda^2 + p\lambda + q = 0 \tag{2.14}$$

determines the solution $y_1(x) = e^{\lambda_1 x}$.
To find all real solutions of the equation, we consider the following three cases:
**Case 1.** The roots $\lambda_1$ and $\lambda_2$ of the characteristic equation

$$\lambda^2 + p\lambda + q = 0, \qquad \Delta = p^2 - 4q > 0, \tag{2.15}$$

are real and distinct.
Then, we have two linearly independent solutions

$$y_1(x) = e^{\lambda_x}, \qquad y_2(x) = e^{\lambda_2 x}$$

Thus, all solutions are in the form of the linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{2.16}$$

for any constants $c_1$ and $c_2$.
Two parameters family of solutions (2.16) is called general solution of homogeneous equation (2.14).

**Example 2.6** *Find the general solution of the equation*

$$y'' - 3y' + 2y = 0$$

**Solution.**The characteristic equation

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 3) = 0, \qquad \Delta = 1 > 0.$$

has two real and distinct roots $\lambda_1 = 1$ and $\lambda_2 = 3$. Then, the linearly independent solutions are:

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = e^{\lambda_2 x}$$

and the general solution is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

for any constants $c_1$ and $c_2$.
**Case 2**. There is a real and double root $\lambda_1 = \lambda_2$ of characteristic equation (2.14). Then, the double root generates two independent solutions

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = xe^{\lambda_1 x}$$

Thus, the general solution of homogeneous equation 2.14) is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

for any constants $c_1$ and $c_2$.

**Example 2.7** *Find the general solution of the equation*

$$y'' - 4y' + 4y = 0, \qquad \Delta = 0.$$

**Solution**. The characteristic equation

$$\lambda^2 - 3\lambda + 2 = (\lambda - 2)^2 = 0$$

has double real root $\lambda_1 = \lambda_2 = 2$. Then, the linearly independent solutions are:

$$y_1(x) = e^{2x}, \qquad y_2(x) = xe^{2x}$$

and the general solution is:

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

for any constants $c_1$ and $c_2$.
**Case 3.** There is a complex root $\lambda_1 = a + ib$ of characteristic equation (2.14). Then, the conjugate $\overline{\lambda}_1 = a - ib$ is also the root of the characteristic equation Using Euler's formula

$$e^{it} = \cos t + i \sin t$$

we find

$$e^{a+ib} = e^a \, e^{ib} = e^a(\cos b + i \sin b)$$

Hence, the complex solutions are

$$e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx)$$

$$e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx)$$

Because the equation is linear one, therefore the real and imaginary parts of the complex solutions are also solutions, that is, the complex root generates two linearly independent solutions

$$y_1(x) = e^{ax} \cos bx, \qquad y_2(x) = e^{ax} \sin bx$$

Then, the general solution of homogeneous equation 2.14) is:

$$y(x) = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$$

for any constants $c_1$ and $c_2$.

**Example 2.8** *Find the general solution of the equation*

$$y'' - 6y' + 25y = 0, \qquad \Delta = -64 < 0.$$

**Solution**. The characteristic equation

$$\lambda^2 - 6\lambda + 25 = 0$$

has complex root $\lambda_1 = \dfrac{6 + 8i}{2} = 3 + 4i, \; a = 3, \; b = 4.$
Then, the linearly independent solutions are:

$$y_1(x) = e^{3x} \cos 4x, \qquad y_2(x) = e^{3x} \sin 4x$$

and the general solution is:

$$y(x) = c_1 e^{3x} \cos 4x + c_2 e^{3x} \sin 4x$$

for any constants $c_1$ and $c_2$.
**Solution of Non-homogeneous equations.** All solutions of the non-homogeneous equation

$$y'' + p\, y' + q\, y = f(x), \qquad a \leq x \leq b, \tag{2.17}$$

are in the of two parameters $c_1$ and $c_2$ family of solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

where $c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the homogeneous equation, when $f(x) = 0$ for all $\in [a, b]$, and $y_p(x)$ is a particular solution of non-homogeneous equation (2.17). Therefore, to find all solutions of the non-homogeneous equation, first, we find the general solution of the homogeneous equation, and then, we find a particular solution $y_p(x)$ of the non-homogeneous equation. The sum of two is the general solution of the non-homogeneous equation.
We shall present two methods for finding a particular solution of a non-homogeneous equation.
- the method of under determined coefficients
- the method of variation of parameters.

**Method of undetermined coefficients**. This method is used to find a particular solution of the non-homogeneous equation. It is applicable to a special class of function $f(x)$ in the right side of the equation.

- When $f(x) = p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial. Assume a particular solution also as a polynomial.
- When $f(x)$ is an exponential function like

  $$f(x) = e^{ax} \quad \text{or} \quad f(x) = p_n(x) e^{ax}$$

  Assume a particular solution also as an exponential function or a combination of a polynomial and an exponential function.
- hen $f(x) = A \sin x + B \cos x$ is a combination of trigonometric functions. Assume a particular solution also as a combination of trigonometric functions.

**Example 2.9** *Find a particular solution of the equation*

$$y'' - 2y' + y = 1 + x^2$$

The right side $f(x) = 1 + x^2$ is the polynomial. So, we predict a particular solution also as a polynomial

$$y_p(x) = a_0 + a_x x + a_2 x^2$$

where the coefficients $a_0,\ a_1\ a_2$ are to be determined.
To find the coefficients $a_0,\ a_1,\ a_2$, we substitute to the equation

$$y_p(x) = a_0 + a_x x + a_2 x^2$$

$$y_p'(x) = a_1 + 2a_2 x$$

$$y_p''(x) = 2a_2$$

to get

$$2a_2 - 2(2a_2 x + a_1) + a_0 + a_1 x + a_2 x^2 = 1 + x^2, \qquad \text{for all} \quad -\infty < x\infty$$

Hence, we have

$$2a_2 - 2a_1 + a_0 = 1$$

$$4a_2 - a_1 = 0$$

$$a_2 = 1$$

The solution is

$$a_2 = 1, \quad a_1 = 4, \quad a_0 = 7,$$

and the particular solution

$$y_p(x) = 7 + 4x + x^2, \qquad -\infty < x < \infty.$$

**Example 2.10** *Find a particular solution of the equation*

$$y'' - y' + 4y = \sin x + \cos x$$

The right side $f(x) = \sin x + \cos x$. So, we predict a particular solution also in the form of the right side

$$y_p(x) = A\sin x + B\cos x$$

where the coefficients $A,\ B$ are to be determined.
To find the coefficients $A,\ B$, we substitute to the equation

$$y_p(x) = A\sin x + B\cos x$$

$$y_p'(x) = A\cos x - B\sin x$$

$$y_p''(x) = -A\sin x - B]cos x$$

to get

$$(-A\sin x - B\cos x) - (A\cos x - B\sin) + 4(A\sin x + B\cos x) = \sin x + \cos x.$$

Hence, we have

$$_(-A+B+4a)\sin x+(-B-A+4B)\cos x = \sin x+\cos x, \qquad \text{for all} \quad -\infty < x < \infty$$

Comparing both sides, we have

$$3A + B = 1$$

$$-A + 3B = 1$$

So that

$$A = \frac{1}{5}, \qquad B = \frac{2}{5}$$

and the particular solution

$$y_p(x) = \frac{1}{5}\sin x + \frac{2}{5}\cos x, \qquad -\infty < x < \infty.$$

**Example 2.11** .

*(a) Find all solutions of the equation*

$$y'' - 2y' - 3y = 2e^x - 10\sin x, \qquad -\infty < x < \infty.$$

*(b) Find the solution of the equation which satisfies the initial value conditions*

$$y(0) = 0, \qquad y'(0) = 1$$

**Solution (a)**. First, we find all solutions of the homogeneous equation

$$y'' - 2y' - 3y = 0, \qquad -\infty < x < \infty.$$

The characteristic equation

$$\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$$

has the real roots $\lambda_1 = -1, \quad \lambda_2 = 3$. Therefore, the linearly independent solutions are

$$y_1(x) = e^{-x}, \qquad y_2(x) = e^{3x}$$

and the general solution of the homogeneous equation is

$$c_1 e^x + c_2 e^{3x}$$

for any constants $c_1$ and $c_2$.
To find a particular solution of the non-homogeneous equation

$$y'' - 2y' - 3y = 2e^x - 10\sin x, \qquad -\infty < x < \infty.$$

We predict a particular solution in the form of the right side

$$y_p(x) = Ae^x + B\sin x + C\cos x$$

where the coefficients $A,\ B,\ C$ are to be determined.
To find the coefficients $A,\ B,\ C$, we substitute to the equation

$$y_p(x) = Ae^x + B\sin x + C\cos x$$

$$y_p'(x) = Ae^x + B\cos x - C\sin x$$

$$y_p''(x) = Ae^x - B\sin x - C\ cosx$$

to get

$$\begin{aligned}(Ae^x - B\sin x - C\cos x) &- 2(Ae^x + B\cos xx - C\sin)\\ &- 3(Ae^x + B\sin x + c\cos x)\\ &= e^x - 10\sin x.\end{aligned}$$

or

$$(-4Ae^x + (-4B + 2C)\sin x - (-4C - 2B)\cos x) = 2e^x - 10\sin x$$

Comparing both sides, we have

$$-4A = 2$$

$$-4B + 2C = -10$$

$$-4C - 2B = 0$$

So that, the solution is

$$A = -\frac{1}{2}, \qquad B = 2, \qquad C = -1$$

and the particular solution

$$y_p(x) = -\frac{1}{2}e^x + 2\sin x - \cos x, \qquad -\infty < x < \infty.$$

All solutions of the non-homogeneous equation are in the two parameters family of solutions

$$y(x) = c_1e^{-x} + c_2e^{3x} - \frac{1}{2}e^x + 2\sin x - \cos x$$

for any constants $c_1$ and $c_2$
**Solution (b)**. By the initial conditions

$$y(0) = c_1 + c_2 - \frac{1}{2} - 1 = 0$$

$$y'(0) = -c_1 + 3c_2 - \frac{1}{2} + 2 = 1$$

Hence, we find

$$c_1 = \frac{5}{4}, \qquad c_2 = \frac{1}{4}$$

and the solution

$$y(x) = \frac{5}{4}e^{-x} + \frac{1}{4}e^{3x} - \frac{1}{2}e^{x} + 2\sin x - \cos x$$

satisfies the initial conditions.

**Method of variation of coefficients**.Let us assume that we have two linearly independent solutions

$$y_1(x) \qquad \text{and} \qquad y_2(x)$$

of the homogeneous equation

$$y''(x) + p\, y'(x) + q\, y(x) = 0, \qquad -\infty < x < \infty$$

Then, all solutions of the homogeneous equation are in the form of the linear combination of two

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

for any constants $c_1$ and $c_2$.

To obtain a particular solution of the non-homogeneous equation

$$y''(x) + p\, y'(x) + q\, y(x) = f(x), \qquad a \le x \le b$$

we apply the method of variation of parameters predicting a particular solution in the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where the functions $v_(x)$ and $v_2(x)$ are to be determined, so that $y_p(x)$ is a particular solution of the non-homogeneous equation.

We put the condition

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0, \qquad \text{for all} \quad a \le x \le b. \tag{2.18}$$

for the functions $v_1(x)$ and $v_2(x)$.

By condition (2.18), we find

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x)$$

and

$$y''(x) = v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)$$

By substitution the above to the equation, we have

$$\begin{aligned} & [v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] \\ + \;\; & p[v_1(x)y_1'(x) + v_2(x)y_2'(x)] \\ + \;\; & q[v_1(x)y_1(x) + v_2(x)y_2(x)] = f(x) \end{aligned}$$

The above can be written as follows:

$$\begin{aligned} & v_1(x)[y_1''(x) + p\, y_1'(x) + q\, y_1(x)] \\ + & \ v_2(x)[y_2''(x) + p\, y_2'(x) + q\, y_2(x)] \\ + & \ [v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = f(x) \end{aligned}$$

Because, $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, therefore

$$y_1''(x) + p\, y_1'(x) + q\, y_1(x) = 0, \quad \text{and} \quad y_2''(x) + p\, y_2'(x) + q\, y_2(x) = 0$$

Hence, we have the following conditions imposed on the functions $v_1(x)$ and $v_2(x)$:

$$\begin{aligned} v_1'(x)y_1(x) + v_2'(x)y_2(x) &= 0 \\ v_1'(x)y_1'(x) + v_2'(x) + v_2'(x)y_2'(x) &= f(x) \end{aligned}$$

The determinant

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

of the matrix

$$A = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

is called Wronskian.
Solving the two equations for $v_1^{'}$ and $v_2^{'}$, we find the solution

$$v_1'(x) = \frac{1}{W[y_1, y_2](x)} \begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix},$$

and

$$v_2'(x) = \frac{1}{W[y_1, y_2](x)} \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix},$$

or

$$v_1'(x) = -\frac{f(x)y_2(x)}{W[y_1, y_2](x)}, \quad \text{and} \quad v_2'(x) = \frac{f(x)y_1(x)}{W[y_1, y_2](x)}$$

Integrating both sides, we find the functions

$$v_1(x) = -\int_a^x \frac{f(s)y_2(s)}{W[y_1, y_2](s)} ds, \quad v_2(x) = \int_a^x \frac{f(s)y_1(s)}{W[y_1, y_2](s)} ds \tag{2.19}$$

Thus, having $v_1(x)$ and $v_2(x)$, we find the particular solution

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x), \qquad a \le x \le b.$$

**Example 2.12** .

*(a) Find a particular solution of the equation*

$$y''(x) - 4y'(x) + 3y(x) = \sin x, \qquad -\infty < x < \infty$$

*(b) Find a solution of the equation which satisfies the initial conditions*

$$y(0) = 1, \qquad y'(0) = 0$$

**Solution (a)**. First, we have to find linearly independent solutions of the homogeneous equation

$$y''(x) - 4y'(x) + 3y(x) = 0, \qquad -\infty < x < \infty$$

The characteristic equation

$$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

has the real and distinct roots $\lambda_1 = 1$ and $\lambda_2 = 3$. Then, the linearly independent solutions are

$$y_1(x) = e^x, \qquad y_2(x) = e^{3x}$$

We predict a particular solution of the non-homogeneous equation in the form

$$y_p(x) = v_1(x)e^x + v_2(x)e^{3x}$$

We impose the condition

$$v_1'(x)e^x + v_2'(x)e^{3x} = 0, \qquad \text{for all} \qquad -\infty < x < \infty.$$

on the functions $v_1(x)$ and $v_2(x)$.
Then, we substitute to the equation

$$\begin{aligned} y_p(x) &= v_1(x)e^x + v_2(x)e^{3x} \\ y_p'(x) &= v_1(x)e^x + v_1'(x)e^x + 3v_2(x)e^{3x} + v_2'(x)e^{3x} \\ &= v_1(x)e^x + 3v_2(x)e^{3x} \\ y_p''(x) &= v_1(x)e^x + v_1'(x)e^x + 3v_2(x)e^{3x} + v_2'(x)e^{3x} \end{aligned}$$

to obtain

$$\begin{aligned} &[v_1(x)e^x + v_1'(x)e^x + 3v_2(x)e^{3x} + v_2'(x)e^{3x}] \\ &-4[v_1(x)e^x + 3v_2(x)e^{3x}] + 3[v_1(x)e^x + v_2(x)e^{3x}] = \sin x \end{aligned}$$

or after simplification

$$v_1'(x)e^x + v_2'(x)e^{3x} = \sin x$$

Thus, the derivatives $v_1'(x)$ and $v_2'(x)$ satisfy two linear equations

$$v_1'(x)e^x + v_2'(x)e^{3x} = 0$$
$$v_1'(x)e^x + v_2'(x)e^{3x} = \sin x$$

The determinant

$$W[y_1, y_2](x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 2e^{4x} > 0$$

is the Wronskian of the matrix

$$A = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix}$$

Solving the system of two equations, we find the solution

$$v_1'(x) = \frac{1}{2}e^{-4x}\begin{vmatrix} 0 & e^{3x} \\ \sin x & 3e^{3x} \end{vmatrix} = -\frac{1}{2}e^{-x}\sin x$$

and

$$v_2'(x) = \frac{1}{2}e^{-4x}\begin{vmatrix} e^x & 0 \\ e^x & \sin x \end{vmatrix} = \frac{1}{2}e^{-3x}\sin x$$

Integrating both sides, we find the functions

$$v_1(x) = -\frac{1}{2}\int_0^x e^{-s}\sin\ s\ ds, \qquad v_2(x) = \frac{1}{2}\int_0^x e^{-3s}\sin\ s\ ds$$

Applying formula of integrating by parts, we evaluate

$$v_1(x) = \frac{1}{4}e^{-x}(\sin x + \cos x - e^x)$$

and

$$v_2(x) = \frac{1}{20}(e^{-3x}(e^{-3x} - \sin x - \cos x)$$

Having the functions $v_1(x)$ and $v_2(x)$, we find the particular solution

$$y_p(x) = v_1(x)e^x + v_2(x)e^{3x}, \qquad -\infty < x < \infty.$$

Let us note that, we can get the same particular solution applying straight forward formulae (2.19).

## 2.5 Exercises, Set 2

**Linear First Order Equation.** Consider the linear differential equation

$$\frac{dy(x)}{dx} + p(x)y = q(x), \qquad a \le x \le b. \tag{2.20}$$

Below, we shall solve the linear differential equation (2.20). First, let us solve the homogeneous linear equation

$$\frac{dy(x)}{dx} + p(x)y = 0, \qquad a \le x \le b. \tag{2.21}$$

when the function $q(x) = 0$ for all $x$.
So, we write the equation in terms of the differentials

$$\frac{dy}{y} = -p(x)dx, \qquad a \le x \le b. \tag{2.22}$$

Integrating both sides, we find

$$\ln|y| = -\int p(x)\, dx + \ln C$$

or

$$y(x) = Ce^{-\int p(x)\, dx}$$

for a constant $C$

**Example 2.13** *Find all solutions of the equation*

$$\frac{dy}{dx} - 2xy = 0, \qquad -\infty < x < \infty.$$

We write the equation in the differentials form

$$\frac{dy}{y} = 2x\, dx, \qquad -\infty < x < \infty.$$

Integrating both sides, we find

$$\ln|y| = x^2 + \ln C$$

Hence, we obtain all solutions

$$y(x) = C\, e^{x^2}$$

for a constant $C$.
We note that all solutions are given in the form of one parameter $C$ family of solutions.

Now, let us solve the non-homogeneous equation. Multiply equation (2.20) by the integrating factor $e^{\int p(x)\,dx}$

$$e^{\int p(x)dx}\frac{dy(x)}{dx} + p(x)e^{\int p(x)dx}y = e^{\int p(x)dx}q(x), \quad a \le x \le b.$$

We note that the left side of the equation is the derivative of the product

$$\frac{d}{dx}[e^{\int p(x)dx}y] = e^{\int p(x)dx}q(x).$$

Integrating both sides, we obtain

$$e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx + C.$$

Hence, the solution

$$y(x) = e^{-\int p(x)dx}[\int e^{\int p(x)dx}q(x)dx + C],$$

or

$$y(x) = e^{-\int p(x)dx}\int e^{\int p(x)dx}q(x)dx + C\ e^{-\int p(x)dx}, \quad a \le x \le b.$$

for a constant $C$.

**Example 2.14** *Find all solutions of the equation*

$$\frac{dy}{dx} + 2y = x, \qquad -\infty < x < \infty$$

We have $p(x) = 2,\ q(x) = x$. Let us multiply the equation by the factor $e^{2x}$, to get

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = x\ e^{2x}, \qquad -\infty < x < \infty$$

Now, the left side of the equation is the derivative of the product

$$\frac{d}{dx}[e^{2x}y] = x\ e^{2x}.$$

Integrating both sides, we find

$$e^{2x}y = \int x\ e^{2x}dx + C$$

Hence, we have the solution

$$y(x) = e^{-2x}\int x\ e^{2x}dx + Ce^{-2x}$$

or

$$y(x) = \frac{1}{4}(2x-1) + Ce^{-2x}, \qquad -\infty < x < \infty$$

for any value of the parameter $C$.

**Question 2.6** *Consider the linear homogeneous differential equation*

$$\frac{dy(x)}{dx} + p(x)y = 0, \qquad a \le x \le b. \tag{2.23}$$

*Show that if* $w(x)$ *and* $v(x)$ *are two solutions of the homogeneous equation, then, for any constants* $c_1$ *and* $c_2$*, the function*

$$y(x) = c_1 w(x) + c_2 v(x)$$

*is also the solution of the homogeneous equation*

**Question 2.7** *Prove that if* $w(x)$ *and* $v(x)$ *are two solutions of the non-homogeneous equation*

$$y'(x) + p(x)y(x) = q(x)$$

*then the difference*

$$y(x) = w(x) - v(x)$$

*is the solution of the homogeneous equation*

$$y'(x) + p(x)y(x) = 0$$

**Question 2.8** *Find all solutions of the equations*

*(a)*

$$\frac{dy}{dx} + 4y = 2x, \qquad -\infty < x < \infty$$

*Answer:* $y(x) = C\,e^{-4x} + \frac{1}{2}(x - \frac{1}{4})$

*(b)*

$$\frac{dy}{dx} + 4xy = 8x, \qquad -\infty < x < \infty$$

*Answer:* $y(x) = 2 + C\,e^{-2x^2}$

**Question 2.9** *Find a continuous solution of the initial value problem*

$$y'(x) + y(x) = f(x), \qquad y(0) = 2$$

*where*

$$f(x) = \begin{cases} x, & -\infty < x \le 1, \\ 2x - 1, & 1 < x < \infty \end{cases}$$

Answer:

$$y(x) = \begin{cases} -1 + x + 3e^{-x}, & -\infty < x \le 1, \\ 2x - 3 + e^{(1-x)} + 3e^{(-x)}, & 1 < x < \infty \end{cases}$$

**Question 2.10** . *Find the solution $y(x)$ of the equation.*

$$y'(x) + 3x^2 y(x) = x^2,$$

*which satisfies the initial value condition $y(0) = 2$.*
*Answer:* $y(x) = \frac{1}{3}(1 + 5e^{-x^2})$

**Question 2.11** *Solve the initial value problem*

$$y' - y = \sin 2x, \qquad -\infty < x < \infty$$

$$y(0 = 1$$

*Answer:*$y(x) = \frac{1}{5}(7e^x - 2\cos 2x \ - \sin 2x)$

**Separable Equations** Let the function $f(x, y) = h(x)g(y)$ be the product of two function $h(x)$ and $g(y)$. So that, each of them is a function of one variable either $x$ or $y$. Then, the separable equation is:

$$\frac{dy}{dx} = h(x)g(y) \tag{2.24}$$

For example, the equation

$$\frac{dy}{dx} = \frac{1 + x^2}{1 + y^2}$$

is separable, since the function

$$f(x, y) = (1 + x^2)\ \frac{1}{1 + y^2}$$

is the product of two functions $h(x) = 1 + x^2$ and $g(y) = \dfrac{1}{1 + y^2}$ and each of them is one variable function either $x$ or $y$.
In order to solve a separable equation, we write equation (2.24) in terms of the differentials

$$\frac{dy}{g(y)} = h(x)\ dx$$

Integrating both sides, left with respect to $y$, and right with respect to $x$, we find the solution $y(x)$ in the implicit form

$$\int \frac{dy}{g(y)} = \int h(x)\ dx$$

In the example, we find the implicit form of the solution

$$\int (1 + y^2) dy = (\int 1 + x^2)\ dx + C$$

So that

$$y + \frac{y^3}{3} = x + \frac{x^3}{3} + C$$

or

$$(y - x) + \frac{1}{3}(y^3 - x^3) = C$$

for any constant $C$.

**Question 2.12** *Solve the following separable equations*

*(a)*

$$\frac{dy}{dx} = -2y \ \tan x$$

*Answer:* $y(x) = C \ \cos^2 x$

*(b)*

$$2x(1 + y)dx + (1 + x^2)dy = 0$$

*Answer:* $y(x) = \dfrac{C}{1 + x^2} - 1$

**Question 2.13** *Find the solution $y(x)$ of the equation*

$$2y \ dx + (2x - 1)dy = 0$$

*which satisfies the initial condition* $y(0) = 1$
*Answer* $y(x) = \dfrac{1}{1 - 2x}$

**Equations with homogeneous** $f(x, y)$. Consider the equation

$$\frac{dy}{dx} = f(x, y)$$

where the function $f(x, y)$ satisfies the condition $f(tx, ty) = f(x, y)$ for real $t$.

**Question 2.14** *Find all solutions of the equation*

*(a)*

$$\frac{dy}{dx} = \frac{x^3}{4x^3 - 3x^2 y}$$

*(b)*

$$(x^2 - 3y^2)dx + 2x \ y \ dy = 0$$

**Second Order Differential Equations** Consider the equation

$$y''(x) + a_1 y'(x) + a_2 y(x) = f(x), \qquad a \leq x \leq b, \tag{2.25}$$

where $a_1, \ a_2$ are given constant coefficients and the right side $f(x)$ is a given continuous function in the interval $a, b]$. Here $y(x)$ is the unknown function of

the variable $x$.
First, we solve the homogeneous equation

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0, \qquad a \leq x \leq b, \tag{2.26}$$

when the right side $f(x) = 0$ for all $x \in [a, b]$.
The roots of the characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

determine the fundamental set of solutions
**Case1.** Let the roots $\lambda_1$ and $\lambda_2$ be real and distinct $(\Delta = a_1^2 - 4a_2 > 0)$
Then, the linearly independent solutions are

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = e^{\lambda_2 x}$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

where $c_1$ and $c_2$ are any constants.

**Example 2.15** *Find all solutions of the equation*

$$y'' - 3y' + 2y = 0$$

The characteristic equation

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0, \qquad (\Delta = 1 > 0)$$

has two real and distinct roots $\lambda_1 = 1$ and $\lambda_2 = 2$.
Thus, the linearly independent solutions are:

$$y_1(x) = e^x, \qquad y_2(x) = e^{2x}$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^x + c_2 e^{2x}$$

for any constants $c_1$ and $c_2$.
**Case 2.** Let the characteristic equation has a double real root $\lambda_1 = \lambda_2$, $(\Delta = a_1^2 - 4a_2 = 0)$.
Then, the linearly independent solutions are

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = xe^{\lambda_1 x}$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

where $c_1$ and $c_2$ are any constants.

**Example 2.16** *Find all solutions of the equation*

$$y'' - 4y' + 4y = 0$$

The characteristic equation

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0, \qquad (\Delta = 0)$$

has two real and distinct roots $\lambda_1 = \lambda_2 = 2$.
Thus, the linearly independent solutions are:

$$y_1(x) = e^{2x}, \qquad y_2(x) = xe^{2x}$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

for any constants $c_1$ and $c_2$.
**Case 3.** Let the characteristic equation has a complex roots $\lambda_1 = \alpha + i\beta$, $\overline{\lambda}_1 = \alpha - i\beta$ , $(\Delta = a_1^2 - 4a_2 < 0)$.
Then, the linearly independent solutions are

$$y_1(x) = e^{\alpha x} \sin \beta x, \qquad y_2(x) = e^{\alpha x} \cos \beta x$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^{\alpha x} \sin \beta x + c_2 e^{\alpha x} \cos \beta x$$

where $c_1$ and $c_2$ are any constants.

**Example 2.17** *Find all solutions of the equation*

$$y'' - y' + y = 0$$

The characteristic equation

$$\lambda^2 - \lambda + 1 = 0, \qquad (\Delta = -3)$$

has complex roots $\lambda_1 = \dfrac{1 + i\sqrt{3}}{2}$, $\overline{\lambda} = \dfrac{1 - i\sqrt{3}}{2}$.
Thus, the linearly independent solutions are:

$$y_1(x) = e^{\frac{x}{2}} \sin \frac{\sqrt{3}}{2} x, \qquad y_2(x) = e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x$$

and all solutions are in form of the linear combination

$$y(x) = c_1 e^{\frac{x}{2}} \sin \frac{\sqrt{3}}{2} x + c_2 e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x$$

for any constants $c_1$ and $c_2$.

**Question 2.15** *Find all solutions of the equations*

*(a)*

$$y'' - 7y' + 12y = 0$$

*(b)*

$$y'' - 6y' + 9y = 0$$

*(c)*

$$y'' - 2y' + 2y = 0$$

**Question 2.16** *Find the solution of the equations which satisfies the indicated initial value conditions*

*(a)*

$$y'' - 6y' + 5y = 0, \qquad y(0) = 0, \;\; y'(0) = 1$$

*(b)*

$$y'' - 4y' + 4y = 0 \qquad y(0) = 1, \;\; y'(0) = 0$$

*(c)*

$$y'' - 2y' + 2y = 0, \qquad y(0) = 0, \;\; y'(0) = 1$$

**Question 2.17** *Find the solution of the non-homogeneous equations which satisfies the indicated initial value conditions*

*(a)*

$$y'' - 6y' + 5y = 1 + x^2, \qquad y(0) = 0, \;\; y'(0) = 1$$

*(b)*

$$y'' - 2y' + 2y = \sin x + \cos x, \quad y(0) = 0, \;\; y'(0) = 0$$

# Chapter 3

# Taylor Polynomials and Taylor Theorem

## 3.1 Taylor Polynomials

Let $f(x)$ be a function n-times continuously differentiable in the interval $[a, b]$. Then, the polynomial

$$\begin{aligned} TL_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \end{aligned}$$

is called *Taylor polynomial of degree* $n$ of the function $f$ about the point $x_0$.[1]
The numbers

$$\frac{f^{(k)}(x_0)}{n!}, \quad k = 0, 1, \ldots, n,$$

are called *Taylor coefficients* of $f$.
The relationship between a function and its Taylor polynomial is given in the following Taylor's theorem:

## 3.2 Taylor Theorem

**Theorem 3.1** *If $f$ is a function $(n + 1)$ times continuously differentiable in the closed interval $[a, b]$, then there exists a point $\xi_x \in (a, b)$ such that*

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}}{n!}(x - x_0)^n + R_{n+1}(\xi_x), \end{aligned}$$

[1] Taylor polynomial about $x_0 = 0$ of $f$ is referred as Maclaurin's polynomial of $f$.

*for all* $x_0, x \in [a,b]$, *where the remainder* $R_{(n+1)}(\xi_x)$ *can be written in the following forms:*
**(a)** *The Lagrange's form:*

$$R_{n+1}(\xi_x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{(n+1)}.$$

*where* $\xi_x$ *is between* $x$ *and* $x_0$.
**(b)** *The Cauchy's form:*

$$R_{n+1}(\xi_x) = \frac{f^{(n+1)}(x_0)}{n!}(x-x_0)(x-\xi_x)^n.$$

**(c)** *The Integral form:*

$$R_{n+1}(\xi_x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

**Proof.** We shall prove Taylor theorem with the remainder $R_{n+1}(\xi_x)$ given in the Lagrange's form (**a**). Let us consider the following auxiliary function:

$$g(t) = f(x) - f(t) - \frac{x-t}{1!}f'(t) - \frac{(x-t)^2}{2!}f''(t) - \cdots \frac{(x-t)^n}{n!}f^{(n)}(t), \quad (3.1)$$

for $x, t \in [a,b]$.
Obviously, the derivative $g'(t)$ exists and

$$\begin{aligned} g'(t) &= -f'(t) + f'(t) - \frac{x-t}{1!}f''(t) \\ &+ \frac{x-t}{1!}f''(t) \cdots - \frac{(x-t)^n}{n!}f^{(n+1)}(t) \\ &= -\frac{(x-t)^n}{n!}f^{(n+1)}(t). \end{aligned}$$

Now, let us consider another auxiliary function

$$G(t) = g(t) - \frac{g(x_0)}{(x-x_0)^{k+1}}(x-t)^{k+1},$$

where $t$ is between $x_0$ and $x$.
This function satisfies Rolle's theorem for any integer k, since we have

$$G(x_0) = G(x) = 0,$$

and $G'(t)$ exists in the open interval $(a,b)$. By the Rolle's theorem, there exists a point $\xi_x$ such that

$$G'(\xi_x) = 0.$$

On the other hand

$$G'(t) = g'(t) + \frac{g(x_0)}{(x-x_0)^{k+1}}(k+1)(x-t)^k.$$

Thus, for $k = n$ and $t = \xi_x$, we have

$$-\frac{(x-\xi_x)^n}{n!}f^{(n+1)}(\xi_x) + g(x_0)(n+1)(x-x_0)^{n+1}(x-\xi_x)^n = 0.$$

Hence

$$g(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

and by (3.1), we obtain the Taylor formula

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ &+ \frac{f^{(n)}}{n!}(x-x_0)^n + R_{n+1}(\xi_x)(x-x_0)^{n+1}, \end{aligned}$$

where the remainder

$$R_{n+1}(\xi_x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1}.$$

### 3.2.1 Examples

**Example 3.1** *Find Taylor polynomial and determine the remainder in the Lagrange's form for the function* $f(x) = e^x$, $-\infty < x < \infty$, *when* $x_0 = 0$.

**Solution.** In order to determine the Taylor polynomial

$$\begin{aligned} TL_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ &+ \frac{f^{(n)}}{n!}(x-x_0)^n, \end{aligned}$$

we shall find the Taylor coefficients of $e^x$. Clearly

$$f^{(n)}(x) = e^x \quad \text{for all } n = 0, 1, 2, \ldots.$$

Thus, Taylor polynomial

$$TL_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

where the Lagrange's remainder

$$R_{n+1}(\xi_x) = \frac{e^{\xi_x}}{(n+1)!}x^{n+1}.$$

for a certain $\xi_x$.

**Example 3.2** *Find the Taylor polynomial for the function* $f(x) = (1+x)^n$, *where* $n$ *is an natural number, about the point* $x_o = 0$.

**Solution.** We have

$$\begin{aligned}
&f^{(0)}(x) = (1+x)^n,\\
&f^{(1)}(x) = n(1+x)^{n-1},\\
&f^{(2)}(x) + n(n-1)(1+x)^{n-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&f^{(n)}(x) = n(n-1)\cdots(n-(n-1)) = n!\\
&f^{(m)}(x) = 0, \quad \text{for} \quad m \geq n+1.
\end{aligned}$$

Hence, at the point $x = 0$,

$$\begin{aligned}
&f^{(0)}(0) = 1\\
&f^{(1)}(0) = n\\
&f^{(2)}(0) = n(n-1)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&f^{(n)}(0) = n!\\
&f^{(m)}(0) = 0, \quad \text{for} \quad m \geq n+1.
\end{aligned}$$

The Taylor polynomial for $f(x) = (1+x)^n$ with $x_o = 0$ is

$$TL_n(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots 1}{n!}x^n,$$

and the remainder is

$$R_n(x) \equiv 0.$$

Hence, by Taylor's theorem, we obtain well known bionomical formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n,$$

**Example 3.3** *Find Taylor polynomial for the function* $f(x) = ln(1+x), \quad 0 \leq x \leq 1$, *about* $x_0 = 0$. *How many terms of the Taylor polynomial are required to approximate the function* $ln(1+x), \quad 0 \leq x \leq 1$, *by its Taylor polynomial with accuracy* $\epsilon = 0.0001$.

**Solution.** In order to obtain Taylor polynomial, we calculate

$$\begin{aligned}
f(x) &= ln(1+x), & f(0) &= 0,\\
f'(x) &= \frac{1}{1+x}, & f'(0) &= 1,\\
f''(x) &= -\frac{1!}{(1+x)^2}, & f''(0) &= -1,\\
f'''(x) &= \frac{2!}{(1+x)^3}, & f'''(0) &= 2!,\\
f^{(4)}(x) &= -\frac{3!}{(1+x)^4}, & f^4(0) &= 3!.
\end{aligned}$$

In general

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \quad \text{and} \quad f^n(0) = (-1)^{n-1}(n-1)!,$$

for $n = 1, 2, \ldots$.
Hence, Taylor polynomial of $ln(1+x)$ at $x_0 = 0$ is

$$TL_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n},$$

where the Lagrange's remainder

$$R_{n+1}(\xi_x) = (-1)^n \frac{x^{n+1}}{(n+1)(1+\xi_x)^{n+1}}, \qquad 0 \le \xi \le 1.$$

The error of approximation

$$ln(1+x) - TL_n(x) = (-1)^n \frac{x^{n+1}}{(n+1)(1+\xi_x)^{n+1}}$$

satisfies the inequality

$$\mid ln(1+x) - TL_n(x) \mid \le \frac{1}{n+1}, \qquad 0 \le x \le 1.$$

The required number of terms of Taylor polynomial $TL_n(x)$ to get the accuracy $\epsilon = 0.0001$ is determined by the following inequality:

$$\frac{1}{n+1} \le 0.0001 \qquad \text{or} \qquad n \ge 9999.$$

We note that the Taylor's series of the function $ln(1+x)$ is slowly convergent. For example, to compute $\ln 2$ with the accuracy $\epsilon$, we need to add about $[\frac{1}{\epsilon}]$ terms. We can compute this sum by the instructions in `Mathematica`

```
N[Sum[(-1)^(n+1)/n,{n,1,9999}]]
```

Then, we obtain $\ln 2 \approx 0.693197$.

**Example 3.4** *Consider the following functions:*

1. $f(x) = \sin x, \quad 0 \le x \le \frac{\pi}{2}$,
2. $f(x) = \cos x, \quad 0 \le x \le \frac{\pi}{2}$.
   (a) *Find Taylor polynomial for the above functions at* $x_0 = 0$.
   (b) *For what value of* $n$ *will Taylor polynomial approximate the above functions correctly upto three decimal places in the interval* $[0, \frac{\pi}{2}]$.

**Solution (a).** In order to find Taylor polynomial

$$\begin{aligned} TL_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ &+ \frac{f^{(n)}(x_0)}{n!}(x-x_))^n, \end{aligned}$$

we shall determine Taylor coefficients

$$\frac{f^{(k)}(0)}{k!}, \qquad k = 0.1, \ldots, n.$$

We have

$$\begin{aligned} f(x) &= sin\ x, & f(0) &= 0, \\ f'(x) &= cos\ x, & f'(0) &= 1, \\ f''(x) &= -sin\ x, & f''(0) &= 0, \\ f'''(x) &= -cos\ x, & f'''(0) &= 0. \end{aligned}$$

In general

$$f^{(n)}(x) = \begin{cases} sin\ x & n = 4k, \quad k = 0, 1, \ldots; \\ cos\ x & n = 4k+1, \quad k = 0.1, \ldots; \\ -sin\ x & n = 4k+2, \quad k = 0, 1, \ldots; \\ -cos\ x & n = 4k+3, \quad k = 0, 1, \ldots; \end{cases}$$

and

$$f^{(n)}(0) = (sin\ 0)^{(n)} = \begin{cases} (-1)^k & n = 2k+1, \quad k = 0, 1, \ldots; \\ 0 & n = 2k, \quad k = 0, 1, \ldots; \end{cases}$$

Thus, Taylor polynomial for $sin\ x$ is

$$TL_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!},$$

where the Lagrange's remainder

$$R_{2n+2}(\xi_x) = \frac{(sin\ \xi_x)^{(2n+2)}}{(2n+2)!}x^{2n+2}.$$

One can obtain the Taylor's polynomial $TL_9(x)$ using the `Mathematica` instruction

```
Normal[Series[Sin[x],{x,0,9}]]
```

In order to get correct three decimal places, we should consider accuracy $\epsilon = 0.0005$, and to choose a smallest $n$ for which the remainder $R_{n+1}(\xi_x)$ satisfies the following inequality

$$|\frac{(sin\xi_x)^{(2n+2)}}{(2n+2)!}x^{2n+2}| \leq \epsilon.$$

Obviously, the above inequality holds if

$$\frac{1}{(2n+2)!}(\frac{\pi}{2})^{2n+2} \leq 0.0005.$$

Hence $n = 4$, so that the Taylor polynomial

$$TL_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

approximates $sin\ x$ in the interval $[0, \frac{\pi}{2}]$ with accuracy upto three decimal places.

**Solution of (b), (ii)** For the function $\cos x$, we find

$$f^{(n)}(x) = (cos\ x)^{(n)} = \begin{cases} cos\ x & \text{for} \quad n = 4k, \quad k = 0, 1, \ldots; \\ -sin\ x & \text{for} \quad n = 4k+1, \quad k = 0, 1, \ldots; \\ -cos\ x & \text{for} \quad n = 4k+2, \quad k = 0, 1, \ldots; \\ sin\ x & \text{for} \quad n = 4k+3, \quad k = 0, 1, \ldots; \end{cases}$$

and

$$cos^{(n)}0 = \begin{cases} (-1)^k & \text{for} \quad n = 2k, \quad k = 0, 1, \ldots; \\ 0 & \text{for} \quad n = 2k+1, \quad k = 0, 1, \ldots; \end{cases}$$

Thus, Taylor polynomial

$$TL_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^2n}{2n!},$$

where the remainder

$$R_{2n+1}(\xi_x) = (-1)^{n+1}\frac{(cos\ \xi_x)^{(2n+1)}}{(2n+1)!}x^{2n+1}.$$

In order to get accuracy of three decimal places, we choose $n = 4$ and the polynomial

$$TL_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

Then, for $n = 4$, we have the following remainder estimate

$$\mid R_{2n+1}(\xi_x) \mid = \mid (-1)^{n+1} \frac{(cos\ \xi_x)^{(2n+1)}}{(2n+1)!} \mid \leq \frac{1}{(2n+1)!} (\frac{\pi}{2})^{(2n+1)} \leq 0.0005.$$

**Question 3.1** *Find the Taylor polynomial $T_3(,f,x)$ and $T_4(g,x)$ of the following functions:*

*(a)*

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2},$$
$$g(x) = \sin \frac{\pi}{2} x + \cos \frac{\pi}{2} x, \qquad 0 \leq x \leq 1$$

*(b) Give an estimate of the errors*

$$E(f) = f(x) - T_3(f,x), \qquad E(g) = g(x) - T_4(g,x), \qquad 0 \leq x \leq 1.$$

*of approximation*

**Question 3.2** *Find Taylor's polynomial $T_3(x)$, $n = 3$ about $x_0 = 0$ for the following functions:*

*(a)*

$$(i) \quad f(x) = \sin x, \qquad (ii) \quad g(x) = \cos x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

*(b) Evaluate an approximate values of $\sin \frac{\pi}{8}$, and $\cos \frac{\pi}{12}$ using the Taylor's polynomials $T_3(x)$. Estimate the errors of approximation*

**Question 3.3** *Find Taylor's polynomial $T_3(x)$, $n = 3$ about $x_0 = 0$ for the following functions:*

*(a)*

$$(i) \quad f(x) = \sqrt{1+x}, \qquad (ii) \quad g(x) = \sqrt{10+x}, \quad 0 \leq x \leq 1.$$

*(b) Evaluate an approximate values of $\sqrt{0.9}$, and $\sqrt{9.9}$ using the Taylor's polynomials $T_3(x)$. Estimate the errors of approximation*

# Chapter 4

# Indeterminate Forms and L'Hopital Rule

## 4.1 Indeterminate Forms

We consider several limits which take the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty,$$

To each of these symbols there corresponds an expression that involves two functions, say $f(x)$ and $g(x)$, and the limit, as $x \to x_o$, or as $x \to \infty$, of the expression considered.
Suppose that the functions $f$ and $g$ are defined in a neighborhood of a given point $x_o$ and suppose that

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} g(x) = 0.$$

Then the limit

$$\lim_{x \to x_o} \frac{f(x)}{g(x)}$$

is said to be of the $\frac{0}{0}$ form.

The symbol $\frac{0}{0}$ is called an indeterminate symbol.
Now we define the first two indeterminate symbols.

**Definition 4.1** *The expression*

$$\frac{f(x)}{g(x)} \tag{4.1}$$

*is of* $\frac{0}{0}$ *form at the point* $x_o$*, if*

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} g(x) = 0;$$

*it is of* $\dfrac{0}{0}$ *form, as* $x \to \infty$*, if*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

**Definition 4.2** *An expression is of* $\dfrac{\infty}{\infty}$ *form at the point* $x_o$*, if*

$$\lim_{x\to x_o} f(x) = \lim_{x\to x_o} g(x) = \infty;$$

*it is of* $\dfrac{\infty}{\infty}$ *form, as* $x \to \infty$*, if*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$$

## 4.2 L'Hôpital's Rule

Suppose that

$$\lim_{x\to x_o} f(x) = 0 \text{ and } \lim_{x\to x_o} g(x) = 0,$$

and suppose that $\lim\limits_{x\to x_o} \dfrac{f'(x)}{g'(x)}$ exists. Then $\lim\limits_{x\to x_o} \dfrac{f(x)}{g(x)}$ also exists and

$$\lim_{x\to x_o} \frac{f(x)}{g(x)} = \lim_{x\to x_o} \frac{f'(x)}{g'(x)}. \tag{4.2}$$

**Example 4.1** *Evaluate*

$$L = \lim_{x->0} \frac{\sin x}{x}$$

**Solution.** We have

$$f(x) = \sin x, \quad f(0) = 0, \quad g(x) = x, \quad g(0) = 0,$$

$$f'(x) = \cos x, \quad f'(0) = 1, \quad g'(x) = 1, \quad g'(0) = 1.$$

By the L'Hopital rule

$$L = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

**Example 4.2** *Evaluate*

$$L = \lim_{x->0} \frac{x^3 - 5x^2 + 6x - 2}{x^5 - 4x^4 + 7x^2 - 9x + 5}$$

**Solution.** We have

$$f(x) = x^3 - 5x^2 + 6x - 2, \qquad f(1) = 0,$$

$$g(x) = x^5 - 4x^4 + 7x^2 - 9x + 5, \quad g(1) = 0,$$

$$f'(x) = 3x^2 - 10x + 6, \qquad f'(1) = -1,$$

$$g'(x) = 5x^4 - 16x^3 + 14x - 9, \qquad g'(1) = -6.$$

Hence, by L'Lopital rule, we obtain

$$L = \lim_{x->1} \frac{x^3 - 5x^2 + 6x - 2}{x^5 - 4x^4 + 7x^2 - 9x + 5} = \lim_{x->1} \frac{3x^2 - 10x + 6}{5x^4 - 16x^3 + 14x - 9} = \frac{1}{6}$$

**Example 4.3** *Using L'Hôpital's Rule to evaluate limits of the* $\frac{0}{0}$ *form at a given point* $x = x_o$.

**(a)** $\lim_{x\to 0} \frac{x + \sin 5x}{x - \sin 5x} = \lim_{x\to 0} \frac{1 + 5\cos 5x}{1 - 5\cos 5x} = -\frac{3}{2}.$

**(b)** $\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5\cos x = 5.$

**(c)** $\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} = -\lim_{x\to 0} \frac{2\sin x \cos x}{2x} = -\lim_{x\to 0} \frac{\sin 2x}{2x} = -1.$

**(d)** $\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{1}{1/\cos^2 x} = \lim_{x\to 0} \cos^2 x = 1.$

**(e)** $\lim_{x\to 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x\to 0} \frac{\sin x}{2\sin x \cos x} = \lim_{x\to 0} \frac{1}{2\cos x} = \frac{1}{2}.$

**(f)** $\lim_{x\to 0} \frac{1 - \cos x^2}{\sin^2 x} = \lim_{x\to 0} \frac{2x \sin x^2}{2\sin x \cos x} = \lim_{x\to 0} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0} \sin x^2 = 1 \cdot 0 = 0.$

**(g)** $\lim_{x\to 0} \frac{2^x - 3^x}{\sin x} = \lim_{x\to 0} \frac{2^x \log 2 - 3^x \log 3}{\cos x} = \log 2 - \log 4 = \log \frac{2}{3}.$

**(h)** $\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^h}{1} = 1;$ **(i)** $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}.$

# Chapter 5

# Improper Integrals

## 5.1 Improper Integrals of the First Kind

We consider improper integrals of the first king with infinite limits of integration

$$\int_{-\infty}^{b} f(x)dx, \quad \int_{a}^{+\infty} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx,$$

when one or both limits of integration are infinite.
In order to evaluate an improper integral of the first kind, we apply the following definition:

**Definition 5.1** *Let $f$ be a continuous function in intervals either $[a, \lambda]$ or $[\lambda, b]$ for every either $\lambda > a$ or $\lambda < b$. If the limits*

$$\lim_{\lambda \to \infty} \int_{a}^{\lambda} f(x)dx, \qquad \lim_{\lambda \to \infty} \int_{-\lambda}^{b} f(x)dx \tag{5.1}$$

*exists, then we say that the integrals*

$$\int_{a}^{\infty} f(x)dx, \quad \int_{-\infty}^{b} f(x)dx$$

*are convergent and we write*

$$\int_{a}^{\infty} f(x)dx = \lim_{\lambda -> \infty} \int_{a}^{\lambda} f(x)dx, \qquad \int_{-\infty}^{b} f(x)dx = \lim_{\lambda > \infty} \int_{-\lambda}^{b} f(x)dx.$$

*If the limits (5.1) do not exist, we say that the improper integrals*

$$\int_{a}^{\infty} f(x)dx, \quad \text{or} \quad \int_{-\infty}^{b} f(x)dx$$

*are divergent.*

A similar approach is used for integrals over the entire line. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as the limit of $\int_{\mu}^{\lambda} f(x)dx$, when $\mu \to -\infty$ and

$\lambda \to +\infty$, in dependently of each other, provided that this limit exists. Equivalently, the improper integral $\int_{-\infty}^{\infty} f(x)dx$ can be expressed as the sum

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx,$$

where $b$ is any real number. Since $b$ can be arbitrarily chosen, in practice we choose $b$ so that the two integrals $\int_{-\infty}^{b} f(x)dx$ and $\int_{b}^{+\infty} f(x)dx$ can be easily examined.

**Example 5.1** *Consider the infinite integral*

$$\int_{1}^{\infty} \frac{1}{x^2}dx.$$

We evaluate the proper in the limits from 1 to $\lambda$

$$\int_{1}^{\lambda} \frac{1}{x^2}dx = (-\frac{1}{x})|_1^{\lambda} = 1 - \frac{1}{\lambda},$$

Then, we find

$$\int_{1}^{\infty} \frac{1}{x^2}dx = \lim_{\lambda -> \infty} \int_{1}^{\lambda} \frac{1}{x^2}dx = \lim_{\lambda -> \infty} (1 - \frac{1}{\lambda}) = 1$$

Thus, the improper integral converges to the limit 1:

**Example 5.2** *Consider the improper integral*

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx.$$

We evaluate the proper integral from 1 to $\lambda$

$$\int_{1}^{\lambda} \frac{1}{\sqrt{x}}dx = 2\sqrt{x}|_1^{\lambda} = 2(\sqrt{\lambda} - 1) \to +\infty, \quad \text{as} \quad \lambda \to +\infty.$$

Thus the integral $\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx$ diverges to $+\infty$ and we write

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx = +\infty.$$

**Example 5.3** *Consider the improper integral*

$$\int_{0}^{\infty} \frac{1}{1+x^2}dx$$

We evaluate the proper integral from 0 to $\lambda$

$$\int_0^\lambda \frac{1}{1+x^2}dx = Arctanx|_0^\lambda = Arctan\lambda$$

Then, we find

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{\lambda->\infty}\int_0^\lambda \frac{1}{1+x^2}dx = \lim_{\lambda->\infty} Arctan\lambda = \frac{\pi}{2}$$

Also, we evaluate the integral

$$\begin{aligned}\int_{-\mu}^{\lambda} \frac{1}{1+x^2}dx &= \arctan\lambda + \arctan\mu|_\mu^\lambda,\\ \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \lim_{\lambda,\mu\to+\infty}(\arctan\lambda + \arctan\mu) = \frac{\pi}{2}+\frac{\pi}{2} = \pi.\end{aligned}$$

**Example 5.4** *Consider the improper integral*

$$\int_1^\infty \frac{dx}{x^\alpha}$$

We evaluate the proper integral from 1 to $\lambda$

$$\int_1^\lambda \frac{dx}{x^\alpha} = \frac{x^{1-\alpha}}{1-\alpha}|_1^\lambda = \frac{\lambda^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}, \qquad \alpha \neq 0.$$

Hence, we find

$$\int_1^\infty \frac{dx}{x^\alpha} = \lim_{\lambda->\infty}\int_1^\lambda \frac{dx}{x^\alpha} = \lim_{\lambda->\infty}\frac{\lambda^{1-\alpha}}{1-\alpha} = \begin{cases} \frac{1}{\alpha-1}, & if \;\; \alpha > 1,\\ \infty, & if \;\; \alpha < 1\end{cases}$$

## 5.2 Improper Integrals of the Second Kind

In this section we consider integrals over a finite interval $[a, b]$, when the integrand $f(x)$ has infinite singularity at some point or points in $[a, b]$. Recall that $f$ has an infinite singularity at the point $x = x_o$, if $f$ is not defined at the point $x_o$ and when one-sided limits, as $x$ approaches $x_o$, are infinite.

**Definition 5.2** *Assume that $f$ is integrable on every interval of the form either $[a, b-\varepsilon]$, or $[a+\varepsilon, b]$, where $0 < \varepsilon < b-a$, or $[a+\varepsilon, b]$, but $f$ has an infinite singularity at the point $x = b$ or $x_0 = a$, that is $\lim_{x\to b-} f(x)$ or $\lim_{x\to a+} f(x)$ is infinite.*

*Then improper integrals are defined as*

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx, \qquad \int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^a f(x)dx,$$

*provided that these limits exist.*

**Example 5.5** *Examining convergence of improper integrals of the second kind.*

**(a)** Consider
$$\int_0^1 \frac{1}{\sqrt{x}}dx.$$
The integrand $f(x) = \dfrac{1}{\sqrt{x}}$ is not bounded on the interval $(0, 1]$ and $\lim_{x\to 0-} f(x) = +\infty$. The function $f(x)$, however, is integrable over every interval $[\varepsilon, 1]$, for $0 < \varepsilon < 1$. We find the proper integral from $\epsilon$ to 1
$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_\varepsilon^1 = 2(1-\sqrt{\varepsilon}) \to 2, \quad \text{as} \quad \varepsilon \to 0+.$$
Therefore, the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges to the limit 2:
$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2.$$
**(b)**. Consider
$$\int_0^1 \frac{1}{x}dx.$$
Similarly to **(a)**, the integrand $f(x) = \dfrac{1}{x}$ is not bounded on the interval $(0, 1]$ and $\lim_{x\to 0-} f(x) = +\infty$. Moreover, the function $f(x)$ is integrable over every interval $[\varepsilon, 1]$, for $0 < \varepsilon < 1$. Now we find the proper integral
$$\int_\varepsilon^1 \frac{1}{x}dx = \ln x\Big|_\varepsilon^1 = -\ln\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0+.$$
Therefore, the improper integral $\int_0^1 \frac{1}{x}dx$ diverges to $+\infty$ and we write
$$\int_0^1 \frac{1}{x}dx = +\infty.$$
**(c)**. Consider the integral
$$\int_0^1 \frac{1}{x^\alpha}dx,$$
where $\alpha$ is any real number. We note that we have already considered this integral in **(a)** with $\alpha = \frac{1}{2}$ and in **(b)** with $\alpha = 1$.

We deal with the infinite singularity at the point $a = 0$, and we note that the function $f(x) = \dfrac{1}{x^\alpha}$ is integrable over every interval $[\varepsilon, 1]$ for $0 < \varepsilon < 1$ and for every real value of $\alpha$.

We find the proper integral

$$\int_{\varepsilon}^{1} \frac{1}{x^{\alpha}} dx = \frac{1}{1-\alpha} x^{(1-\alpha)} \Big|_{\varepsilon}^{1} = \frac{1}{1-\alpha} \left(1 - \varepsilon^{1-\alpha}\right),$$

provided $\alpha \neq 1$. Thus

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{1}{x^{\alpha}} dx = \begin{cases} \dfrac{1}{1-\alpha} & \text{if } \alpha < 1 \\ +\infty & \text{if } \alpha > 1. \end{cases}$$

Combining the above with the result obtained in **(b)**, we conclude that the integral

$$\int_{0}^{1} \frac{1}{x^{\alpha}} dx$$

converges for $\alpha < 1$ and diverges for $\alpha \geq 1$. If $\alpha < 1$ then

$$\int_{0}^{1} \frac{1}{x^{\alpha}} dx = \frac{1}{1-\alpha}.$$

**(d)**. Let $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ and consider the integral

$$\int_{0}^{1} f(x) dx = \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx.$$

We note that $\lim_{x \to 1-} f(x) = +\infty$, so that $f(x)$, defined for $|x| < 1$, is not bounded on the interval $[0, 1)$.
The integral has a singularity point at $x = 1$, but $f(x)$ is integrable over any interval $[0, 1-\varepsilon]$, where $0 < \varepsilon < 1$. We find the proper integral

$$\int_{0}^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \Big|_{0}^{1-\varepsilon} = \arcsin(1-\varepsilon) \to \frac{\pi}{2}, \quad \text{as} \quad \varepsilon \to 0+.$$

Hence $\int_{0}^{1} f(x) dx$ converges to $\pi/2$:

$$\int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx = \lim_{\varepsilon \to 0+} \int_{0}^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2}.$$

**(e)**. Consider the integral

$$\int_{0}^{1} \frac{1}{\sqrt{x(1-x)}} dx.$$

It has two singularity points: at $x = 0$ and at $x = 1$.
We find the proper integral

$$\int_{\varepsilon'}^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}} dx = \arcsin(2x-1) \Big|_{\varepsilon'}^{1-\varepsilon} = \arcsin(1-2\varepsilon) - \arcsin(2\varepsilon'-1).$$

Thus

$$\begin{aligned}\lim_{\varepsilon,\varepsilon'\to 0+}\int_{\varepsilon'}^{1-\varepsilon}\frac{1}{\sqrt{x(1-x)}}dx &= \lim_{\varepsilon\to 0+}\arcsin(1-2\varepsilon)-\lim_{\varepsilon'\to 0+}\arcsin(2\varepsilon'-1)\\ &= \arcsin 1-\arcsin(-1)\\ &= \frac{\pi}{2}-(-\frac{\pi}{2})=\pi.\end{aligned}$$

Therefore we conclude that the improper integral

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx$$

converges to $\pi$:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx = \lim_{\varepsilon,\varepsilon'\to 0+}\int_{\varepsilon'}^{1-\varepsilon}\frac{1}{\sqrt{x(1-x)}}dx = \pi.$$

**(f)**. Consider the integral

$$\int_0^1 \frac{1}{1-x}dx,$$

that has a singularity point at $x=1$:
$\lim_{x\to 1-}\frac{1}{1-x}=+\infty$. We find the proper integral

$$\int_0^{1-\varepsilon}\frac{1}{1-x}dx = -\ln(1-x)\Big|_0^{1-\varepsilon} = -\ln\varepsilon \to +\infty \quad \text{as} \quad \varepsilon\to 0+\,.$$

Therefore, $\int_0^1 \frac{1}{1-x}dx$ diverges to $+\infty$:

$$\int_0^1 \frac{1}{1-x}dx = +\infty.$$

## 5.3 Comparison Tests for Integrals

5.3 Comparison Tests for Integrals

**Theorem 5.1 Comparison Test I**

*Suppose that $f$ and $g$ are defined on the interval $[a,+\infty)$ and integrable on $[a,\lambda]$ for every $\lambda > a$.*
*If*

$$0\le f(x)\le g(x),$$

*for all $x\in[a,+\infty)$, then*
**(i)** $\int_a^{+\infty} f(x)dx$ *converges if* $\int_a^{+\infty} g(x)dx$ *converges;*
**(ii)** $\int_a^{+\infty} g(x)dx$ *diverges if* $\int_a^{+\infty} f(x)dx$ *diverges.*

**Proof.** Since $0 < f(x) < g(x)$, for $x \geq a$, we get

$$0 \leq \int_a^\lambda f(x)dx \leq \int_a^\lambda g(x)dx, \quad x \geq a,$$

and each integral is monotone increasing function of $\lambda$.

Hence, if $\int_a^\lambda g(x)dx$ converges, $\int_a^\lambda f(x)dx$ is bounded above and so it converges.

If $\int_a^\lambda f(x)dx$ diverges then $\int_a^\lambda g(x)dx$ is unbounded and hence diverges.

**Example 5.6** *Does* $\int_0^{+\infty} \frac{1}{e^x+3}dx$ *converge?*

**Solution.** Let $f(x) = e^{-x}$ and $g(x) = \frac{1}{e^x+3}$, for $x \in [0, +\infty)$. We have

$$0 < \frac{1}{e^x+3} < \frac{1}{e^x} = e^{-x}, \quad x \in [0, +\infty),$$

and both functions, $f$ and $g$, are integrable on $[0, \lambda]$ for every $\lambda > 0$. Thus the hypotheses of the Comparison Test I are satisfied. Now,

$$\begin{aligned}\int_0^{+\infty} f(x)dx &= \int_0^{+\infty} e^{-x}dx \\ &= -\lim_{\lambda\to+\infty} e^{-x} \\ &= -\lim_{\lambda\to+\infty} e^{-x}|_0^\lambda \\ &= -\lim_{\lambda\to+\infty} (1 - e^{-\lambda}) = 1.\end{aligned}$$

Therefore, by the Comparison Test, the improper integral

$$\int_0^{+\infty} \frac{1}{e^x+3}dx$$

converges.
An analogous comparison test holds for improper integrals of the second kind. We leave its formulation to the reader. The following example illustrates the point.

**Example 5.7** *Does* $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ *converge?*

**Solution.** We have

$$\frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}, \quad x \in (0, 1],$$

and

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to0+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to0+} 2\sqrt{x}\Big|_\varepsilon^1 = \lim_{\varepsilon\to0+} (2 - 2\sqrt{\varepsilon}) = 2.$$

Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, by the Comparison Test, we conclude that

$$\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$$

converges.

**Example 5.8** *Examining convergence of an improper integral of the third kind.*

Consider the infinite integral $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ and note that the integrand has a singular point at $x = 0$. Thus $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ is an improper integral of the third kind.

We can write

$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx = \int_0^b \frac{1}{x^2+\sqrt{x}}dx + \int_b^\infty \frac{1}{x^2+\sqrt{x}}dx,$$

where the point $x = b$ for splitting up the interval of integration can be chosen quite arbitrarily.

Let $b = 1$. We examine separately convergence of each of

$$\int_0^1 \frac{1}{x^2+\sqrt{x}}dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^2+\sqrt{x}}dx.$$

The integral $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ converges by Example 5.7.

The integral $\int_1^\infty \frac{1}{x^2+\sqrt{x}}dx$ converges, by the Comparison Test, because

$\frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}, \; x \in [1,\infty)$ and $\int_1^\infty \frac{1}{x^2}dx$ converges.

Therefore, we conclude that the integral

$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$$

converges.

**Example 5.9** *Is the function* $f(t) = t^{x-1}e^{-t}$*, where* $x$ *is a real number, integrable over the interval* $[0,1]$*?*

**Solution.** We shall consider separately the three cases: $x \le 0$, $0 < x < 1$, and $x \ge 1$. Refer to Figure **??** to see graphs of $f(t)$, $0 < t < 1$, for selected values of $x$.

**Case 1:** $x \le 0$.

On the interval $[0,1]$ we have $e^t < 3$, so $t^{x-1}e^{-t} > \frac{1}{3}t^{x-1}$. By Example 5.5(b),

$\int_0^1 t^{x-1}dt$ diverges, since $x-1 \leq -1$. Hence $\int_0^1 t^{x-1}e^{-t}dt$ diverges for $x \leq 0$.
**Case 2:** $0 < x < 1$.
If $0 < x < 1$, then the function $f(t) = t^{x-1}e^{-t}$ has an infinite singularity at the point $t = 0$ and $\int_0^1 t^{x-1}e^{-t}dt$ is an improper integral of the second kind.
If $t \geq 0$ then $0 < e^{-t} \leq 1$ and $t^{x-1}e^{-t} \leq t^{x-1}$. Now, the improper integral $\int_0^1 t^{x-1}dt$ converges for all values of $x$ such that $0 < 1-x < 1$ or $0 < x < 1$.
By the Comparison Test, therefore, we conclude that the improper integral $\int_0^1 t^{x-1}e^{-t}dt$ converges for $0 < x < 1$.
**Case 3:** $x \geq 1$.
The function $f(t)$ is continuous for $t \in [0,1]$ and, therefore, integrable. Hence $t^{x-1}e^{-t}dt$ exists for $x \geq 1$.
Therefore, the function $f(t) = t^{x-1}e^{-t}$ is integrable over the interval $[0,1]$, provided that $x > 0$, but not integrable if $x \leq 0$.

**Example 5.10** *Is the function* $f(t) = t^{x-1}e^{-t}$, *where* $x > 0$, *integrable on the interval* $[1, +\infty)$*?*

**Solution.** The integral $\int_1^{+\infty} t^{x-1}e^{-t}dt$ is an improper integral of the first kind. We shall prove that it converges by comparing the integrand $f(x)$ with the function $g(t) = t^{-2}$. Now, the improper integral $\int_{t_o}^{+\infty} t^{-2}dt$ converges.
Hence, by the Comparison Test, $\int_{t_o}^{+\infty} e^{-t}t^{x-1}dt$ converges.
Since $f(t) = e^{-t}t^{x-1}$ is integrable on any interval of the form $[1, t_o]$, we conclude that

$$\int_1^{+\infty} t^{x-1}e^{-t}dt = \int_1^{t_o} t^{x-1}e^{-t}dt + \int_{t_o}^{+\infty} t^{x-1}e^{-t}dt$$

converges, when $x > 0$.
Therefore, the function $f(t) = t^{x-1}e^{-t}$, where $x > 0$, is integrable on the interval $[1, +\infty)$.

# Chapter 6

# Sequences and Series

## 6.1 Sequences

Let us begin with the definition

**Definition 6.1** *A sequence of real numbers is a real-valued function $f$ whose domain is the set of natural numbers $N = \{1, 2, ..., \}$, i.e.*

$$f : N -> R, \quad R \, is \, the \, set \, of \, real \, numbers$$

The function $f$ which defines a sequence is a rule that assigns to each natural number $n$ a unique real value, normally denoted by

$$f(n) = a_n, \;\; n = 1, 2, \ldots$$

The number $a_n$ is called the $n$-th term of the sequence and the corresponding sequence is denoted by the symbol

$$\{a_n\} = \{a_1, \; a_2, \; a_3, \; a_4, \; \ldots\}.$$

We will find it convenient to use the notations $\{b_n\}, \{c_n\}, \{d_n\}$, and $\{x_n\}, \{y_n\}$, etc., in addition to $\{a_n\}$, to denote sequences, especially when we deal with two or more sequences at a time.

**Example 6.1** *Defining sequences:*

**(a)** *The function $f(n) = \dfrac{n}{n+1}, n \in N$, defines a sequence with $n$-th term $a_n = \dfrac{n}{n+1}$, so that*

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, \quad a_2 = \frac{2}{2+1} = \frac{2}{3}, \quad a_3 = \frac{3}{3+1} = \frac{3}{4},$$

*and so on. We have*

$$\{a_n\} = \{\frac{1}{2}, \;\; \frac{2}{3}, \;\; \frac{3}{4}, \;\; \frac{4}{5}, \;\; \ldots, \;\; \frac{n}{n+1}, \;\; \ldots\}.$$

**(b)** *The function* $f(n) = \dfrac{(-1)^n}{n^2}$, $n \in N$, *defines the sequence*

$$\{\ b_n\ \} = \{-1,\ \frac{1}{4},\ -\frac{1}{9},\ \frac{1}{16},\ -\frac{1}{25},\ \frac{1}{36},\ \dots\ \frac{(-1)^n}{n^2},\ \dots\ \}.$$

**(c)** *The function*

$$f(n) = c_n = (-1)^n \frac{n^2}{n+2},\ \ n \in N,$$

*defines the sequence*

$$\{c_n\} = \{-\frac{1}{3},\ \frac{4}{4},\ -\frac{9}{5},\ \frac{16}{6},\ -\frac{25}{7},\ \dots,\ (-1)^n \frac{n^2}{n+2},\ \dots\}$$

### 6.1.1 Definition of Convergence

Let us state the intuitive definition of convergence of a sequence $\{a_n\},\ n = 1, 2, ...;$

**Definition 6.2** *If* $a_n$ *approaches a number* $a$, *when* $n$ *approaches* $\infty$, *that is* $a_n -> a$ *when* $n -> \infty$, *so that,* $a$ *is the limit of the sequence* $\{a_n\},\ n = 1, 2, ...;$. *Then, the sequence is convergent to the limit* $a$
*In symbols, we write*

$$\lim_{n->\infty} a_n = a$$

*Otherwise, the sequence* $\{a_n\},\ n = 1, 2, ...;$ *is divergent.*

Thus, as $n$ increase, $a_n$ gets arbitrarily close to $a$.

**Theorem 6.1** *If the limit* $\lim_{n\to\infty} a_n = a$ *exists, then it is unique.*

**Example 6.2** *Showing that* $\lim_{n\to\infty} a_n = a$ *directly from the definition.*

**(a)** Consider the sequence

$$a_n = \frac{n}{n+1}$$

The terms

$$a_1 = \frac{1}{2},$$

$$a_2 = \frac{2}{3},$$

$$a_3 = \frac{3}{4},$$

$$a_4 = \frac{4}{5}$$

$$\vdots$$

$$a_n = 1 - \frac{1}{n+1}, ...$$

Hence, $a_n = \frac{n}{n+1} = 1 - \frac{1}{n+1} -> 1$ when $n -> \infty$, since $\frac{1}{n+1} -> 0$, when $n -> \infty$.
**(b)** Clearly

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$$

If $a_n$ tends to the limit $+\infty$ then we write

$$\lim_{n\to\infty} a_n = +\infty.$$

If $a_n$ has limit $-\infty$ then we write

$$\lim_{n\to\infty} a_n = -\infty.$$

For example the sequence $a_n = n^2,\ n = 1, 2, ...;$ has the infinite limit

$$\lim_{n->\infty} a_n = \lim_{n->\infty} n^2 = \infty$$

Intuitively, $\lim_{n\to\infty} a_n = +\infty$ means that $a_n$ increase without bound, as $n$ increase, whereas $\lim_{n\to\infty} a_n = -\infty$ means that $a_n$ decreases without bound, as $n$ increase.
If $\lim_{n\to\infty} a_n = +\infty$ then with any positive value of $M$, no matter how large it is, we can find an $N$ such that all terms $a_n \geq M$, for $n > N$, are greater than $M$. Similarly, if $\lim_{n\to\infty} a_n = -\infty$, then for any $M > 0$ there exists $N$ such that all terms $a_n \leq -M$, for $n > N$, are less than $-M$.
Infinite limits of the sequence.

$$\lim_{n\to\infty} \frac{n^2}{n+1} = +\infty.$$

Let $M$ be any positive number. Following definition, we are to find $N$ such that $a_n > M$ for all $n > N$. We have

$$a_n = \frac{n^2}{n+1} > \frac{n^2}{2n} = \frac{n}{2} > M, \quad \text{provided} \quad n > 2M,$$

so we can take $N = [2M]$.

### 6.1.2 Bounded Sequences

We consider bounded sequences in the sense of the following definitions:

- A sequence $\{a_n\},\ n = 1, 2, ...$ is bounded if there are two numbers $m$ and $M$ such that

  $$m \leq a_n \leq M$$

  for all $n = 1, 2, ...;$

- A sequence $\{a_n\},\ n = 1, 2, ...$ is bounded above if there is a numbers $M$ such that
$$a_n \leq M$$
for all $n = 1, 2, ...;$

- A sequence $\{a_n\},\ n = 1, 2, ...$ is bounded below if there is a numbers $m$ such that
$$m \leq a_n$$
for all $n = 1, 2, ...;$

The following theorem holds:

**Theorem 6.2** *If the limit*
$$\lim_{n->\infty} a_n,$$
*exists, then the sequence* $\{a_n\}$ *is bounded.*

**Proof**. Let the sequence $\{a_n\}, n = 1, 2, ...$ be convergent to the limit $a$. Then almost all terms of the sequence lie in the neighborhood of the limit $a$. That is, all exempt a finite number of terms, say

$$m = min\{a_1, a_2, ..., a_k\} \leq a_n \leq M = max\{a_1, a_2, ..., a_k\}$$

for a certain $k$. But, all remaining terms satisfy the inequality

$$a - \epsilon \leq a_{k+1}, a_{k+2}, ..., a_n, ... \leq a + \epsilon$$

Hence, all terms of the sequence are between $m_0 = \min\{a - \epsilon, m\}$ and $M_0 = \max\{a + \epsilon, M\}$, that is
$$m_0 \leq a_n \leq M_0,$$
for all $n = 1, 2, ...;$

### 6.1.3 The Algebra of Limits

It is clearly not always straightforward to use the definition of convergence to prove that a sequence $\{a_n\}$ converges to a known limit $a$. Moreover, if the limit $a$ is not known, then the definition of convergence may not help in determining $a$.

Now we are going to introduce some useful results that enable us to evaluate limits of quite complicated sequences without appealing to the definition of convergence.

The following theorem can be used to evaluate the limits of sequences that arise by applying the arithmetic operations of addition, multiplication, and division on convergent sequences with known limits.

**Theorem 6.3** *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences and let* $c$ *be a real number. Then the sequences*

$$\{ca_n\}, \quad \{a_n + b_n\}, \quad \{a_n b_n\}$$

*are convergent and the following rules apply.*

**(i)** *Scalar product rule:*
$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n,$

**(ii)** *Sum rule:*
$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$

**(iii)** *Product rule:*
$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$

**(iv)** *Quotient rule:*
*If* $b_n \neq 0,$ for $n = 1, 2, \ldots,$ *so that the sequence* $\{\frac{a_n}{b_n}\}$ *is defined, and if* $\lim_{n\to\infty} b_n \neq 0$*, then the sequence* $\{\frac{a_n}{b_n}\}$ *converges and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$

### 6.1.4 The Squeeze Theorem for Sequences

When examining the convergence of a given sequence $\{a_n\}$, quite often it is possible to find two sequences, say $\{x_n\}$ and $\{y_n\}$, such that

$$x_n \leq a_n \leq y_n, \text{for } \ n > M \in \ N,$$

so that, eventually, all terms of $\{a_n\}$ are "squeezed" between the corresponding terms of $\{x_n\}$ and $\{y_n\}$. If $\{x_n\}$ and $\{y_n\}$ converge to the same limit $l$, then the sequence $\{a_n\}$ must converge to the limit $l$.

**Theorem 6.4 Squeeze theorem**
*Suppose that*

$$x_n \leq a_n \leq y_n, \text{for } \quad n > M \in N \tag{6.1}$$

*and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l. \tag{6.2}$$

*Then*

$$\lim_{n\to\infty} a_n = l.$$

**Example 6.3** *Prove that* $\lim_{n->\infty} \sqrt[n]{a} = 1, \;$ for $\; a > 0.$

**Solution.** We consider two cases.
**Case 1.** $a \geq 1.$

If $a \geq 1$ then $\sqrt[n]{a} \geq 1$ and $\sqrt[n]{a} = 1 + d_n$, where $d_n \geq 0$. Thus, by the Bernoulli inequality,

$$a = (1 + d_n)^n \geq 1 + nd_n \quad n \geq 2.$$

Since

$$\lim_{n\to\infty} \frac{a-1}{n} = 0,$$

by the Squeeze Theorem, it follows that

$$\lim_{n\to\infty} d_n = 0$$

and consequently

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 + \lim_{n\to\infty} d_n = 1.$$

**Case 2.** $0 < a < 1$.

If $0 < a < 1$, then $a = 1/b$, where $b > 0$, and

$$\lim_{n\to\infty} \sqrt[n]{b} = 1,$$

which implies

$$\lim_{n\to\infty} \sqrt[n]{a} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{b}} = 1. \quad \blacksquare$$

**Example 6.4** *Prove that* $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

**Solution.** We note that $\sqrt[n]{n} > 1$, when $n > 1$, so that we can write

$$\sqrt[n]{n} = 1 + d_n, \quad \textit{where} \;\; d_n > 0$$

and we have

$$n = (1 + d_n)^n = 1 + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n > \binom{n}{2} d_n^2.$$

Thus

$$n > \binom{n}{2} d_n^2 = \frac{n(n-1)}{2} d_n^2$$

which implies that

$$0 < d_n < \sqrt{\frac{2}{n-1}}, \quad n = 2, 3, \ldots,$$

Hence $\lim\limits_{n\to\infty} d_n = 0$ and

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (1 + d_n) = 1.$$

**Example 6.5** *Find the limit*

$$\lim_{n->\infty} \sqrt[n]{2^n + 3^n}$$

Firstly, we show, by squeeze theorem, that the sequence

$$a_n = \sqrt[n]{2^n + 3^n}, \quad n = 1, 2, ...$$

is convergent.
We have the inequality

$$3 < \sqrt[n]{2^n + 3^n} < 3\sqrt[n]{2}, \qquad n = 1, 2, ....$$

Because $\lim \sqrt[n]{2} = 1$ therefore, by the squeeze theorem, the sequence $a_n = \sqrt[n]{2^n + 3^n}, \ \ n = 1, 2, ...$ is convergent and its limit

$$\lim_{n->\infty} \sqrt[n]{2^n + 3^n} = 3$$

### 6.1.5 Monotone Sequences

We consider monotone sequences in the following sense:

- the sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all $n = 1, 2, ...;$
- the sequence $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$ for all $n = 1, 2, ...;$

The following theorem holds:

**Theorem 6.5** .

- *If the sequence $\{a_n\}$ is increasing and bounded above, then it is convergent sequence.*
- *If the sequence $\{a_n\}$ is decreasing and bounded below, then it is convergent sequence.*

**Proof**. By the assumption

$$a_n \leq \sup_{n \in N} a_n = M$$

for all $n = 1, 2, ...;$.
But, the sequence is increasing, therefore, for $\epsilon > 0$ all terms satisfy inequality

$$M - \epsilon \leq a_n \leq M,$$

starting from $n \geq N_\epsilon$. It means that the sequence converges and $a = M$ is the limit.
Similarly, By the assumption

$$a_n \geq \inf_{n \in N} a_n = m$$

for all $n = 1, 2, ...;$.
But, the sequence is decreasing, therefore, for $\epsilon > 0$ all terms satisfy inequality

$$m + \epsilon \geq a_n \geq m,$$

starting from $n \geq N_\epsilon$. It means that the sequence converges and $a = m$ is the limit.

**Example 6.6** *Let $a > b > 0$ be two given real numbers. Consider the sequences of arithmetic and geometric averages*

$$\begin{array}{ll} a_1 = \dfrac{a+b}{2} & b_1 = \sqrt{a\,b} \\ a_2 = \dfrac{a_1+b_1}{2} & b_2 = \sqrt{a_1\,b_1} \\ a_3 = \dfrac{a_2+b_2}{2} & b_3 = \sqrt{a_2\,b_2} \\ \dots\dots\dots\dots & \dots\dots\dots\dots\dots \\ a_{n+1} = \dfrac{a_n+b_n}{2} & b_{n+1} = \sqrt{a_n\,b_n}, \quad n = 1, 2, \dots \end{array}$$

We note that

$$a > a_1 > b_1 > b$$

Indeed, we have

$$\frac{1}{2}(\sqrt{a} - \sqrt{b})^2 = \frac{1}{2}(a+b) - \sqrt{a\,b} > 0$$

Hence

$$a_1 = \frac{a+b}{2} > \sqrt{a\,b} = b_1 > b. \tag{6.3}$$

Similarly, having $a_n$ and $b_n$, we define

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n\,b_n}$$

and we apply inequality (6.3) for $a_n, a_{n+1}, b_n$ and $b_{n+1}$ to obtain

$$a_n > a_{n+1} > b_{n+1} > b_n \tag{6.4}$$

Then, the first sequence $\{a_n\}$ is decreasing and the second sequence $\{b_n\}$ is increasing. But, both sequences are bounded, by $a$ and $b$, since

$$a > a_n > b_n > b$$

for all $n = 1, 2, \dots$
By the theorem both sequences $\{a_n\}$ and $\{b_n\}$ are convergent.
Let the limits

$$\alpha = \lim_{n->\infty} a_n, \qquad \beta = \lim_{n->\infty} b_n.$$

From the equality

$$a_{n+1} = \frac{a_n + b_n}{2}$$

we find

$$\alpha = \frac{\alpha + \beta}{2}, \qquad \alpha = \beta$$

**Example 6.7** *Consider the sequence given by the recursive formula*

$$a_1 = \sqrt{2}, \quad x_2 = \sqrt{2+\sqrt{2}}, \quad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

*In general*

$$a_{n+1} = \sqrt{2+a_n}, \quad n = 1, 2, ....$$

Clearly, the sequence is increasing, since

$$a_{n+1} = \sqrt{2+a_n} = \sqrt{2+\sqrt{2+a_{n-1}}} > \sqrt{2+a_{n-1}} = a_n$$

for $n = 1, 2, ...$
Also, the sequence is upper bounded by $1+\sqrt{2}$, because

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2+\sqrt{2}} < 1+\sqrt{2}$$

and by mathematical induction, if $a_n < 1+\sqrt{2}$ then

$$a_{n+1} = \sqrt{2+\sqrt{a_n}} < \sqrt{2+\sqrt{2}+1} < 1+\sqrt{2}$$

for $n = 1, 2, ....$
By the theorem, the sequence is convergent.
Let

$$\alpha = \lim_{n->\infty} a_n$$

Then, we find

$$\alpha = \lim_{n->\infty} a_{n+1} = \sqrt{2+\lim_{n->\infty} a_n} = \sqrt{2+\alpha}$$

Hence, $\alpha$ satisfies the quadratic equation

$$\alpha^2 = 2+\alpha$$

Then the limit $\alpha = 2$. let us note that the negative root $\alpha = -1$ of the quadratic equation cannot be the limit.

### 6.1.6 The Number e

One of the fundamental constants in Mathematical Analysis is the number

$$e = 2,7182818284\ldots.$$

It can be defined as a limit of an increasing sequence.
Let

$$a_n = (1+\frac{1}{n})^n, \quad n = 1, 2, \ldots$$

We shall show that $\{a_n\}$ is strictly increasing and bounded above. So that, by virtue of theorem, $\{a_n\}$ is convergent.
By the Bernoulli inequality,

$$(1-\frac{1}{n^2})^n > 1-\frac{1}{n} \text{ for } n>1.$$

We have

$$\begin{aligned}(1-\frac{1}{n^2})^n > 1-\frac{1}{n} \quad &\textit{implies} \quad (1+\frac{1}{n})^n(1-\frac{1}{n})^n > 1-\frac{1}{n} \\ &\textit{implies} \quad (1+\frac{1}{n})^n(1-\frac{1}{n})^{n-1} > 1,\end{aligned}$$

for all $n = 1, 2, ...$
Therefore

$$\left(1+\frac{1}{n}\right)^n > \left(\frac{1}{1-\frac{1}{n}}\right)^{n-1} = \left(\frac{n}{n-1}\right)^{n-1}$$

and we get

$$a_n = (1+\frac{1}{n})^n > (\frac{n}{n-1})^{n-1} = (1+\frac{1}{n-1})^{n-1} = a_{n-1}, \text{for all } n = 1, 2, ..$$

In order to show that the sequence is bounded above, we apply the binomial expansion, for $n > 2$,

$$a_n = (1+\frac{1}{n})^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}(\frac{1}{n})^k = 1+1+\sum_{k=2}^{n}\binom{n}{k}(\frac{1}{n})^k = 2+\sum_{k=2}^{n}\alpha_k.$$

The $k$-th term of the sum $\sum \alpha_k$ can be written as

$$\begin{aligned}\alpha_k = \binom{n}{k}(\frac{1}{n})^k &= \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{k!}(\frac{1}{n})^k \\ &= \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-k+2}{n}\frac{n-k+1}{n}\frac{1}{k!} \\ &= \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}).\end{aligned}$$

So that

$$\alpha_k < \frac{1}{k!} = \frac{1}{1\times 2\times 3\times\cdots\times k} < \frac{1}{1\times 2\times 2\times\cdots\times 2} = \frac{1}{2^{k-1}},$$

for $k = 2, 3, \ldots, n$. Therefore

$$\begin{aligned}a_n &= 1+1+\sum_{k=2}^{n}\alpha_k < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} \\ &= 1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1+2(1-(\frac{1}{2})^n) < 3,\end{aligned}$$

for all $n = 1, 2, \ldots$.
By the theorem on monotone and bounded sequences, the sequence

$$a_n = (1 + \frac{1}{n})^n, \quad n = 1, 2, ...;$$

is convergent and its limit

$$\lim_{n->\infty} (1 + \frac{1}{n})^n = e = 2.7182818284...$$

## 6.2 Infinite Series

Let us begin with the definition

**Definition 6.3** *Let $\{a_k\}$, $k = 0, 1, 2, \ldots$, be a given sequence of real numbers. Consider the sequence $\{S_n\}$ defined as the sum of the first $n+1$ terms of $\{a_k\}$:*

$$S_n = a_o + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k. \tag{6.5}$$

*If $\{S_n\}$ converges to the limit $S$,*

$$\lim_{n->\infty} S_n = S,$$

*then we define $\sum_{k=0}^{\infty} a_k$ to be $S$:*

$$S = \sum_{k=0}^{\infty} a_k \tag{6.6}$$

*and call $S$ the sum of the infinite series The series is then said to be convergent; otherwise it is said to be divergent. The sum $S_n$ defined by is called the n-th partial sum of the infinite series*

**Example 6.8** *Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.*

**Solution.** We have

$$a_n = \frac{1}{n(n+1)}.$$

and

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

Hence, we obtain

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

So that $S_n = 1 - \dfrac{1}{n+1} -> 1$ when $n -> \infty$

Thus, the given series converges and its sum $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$.

The necessary condition for convergence of a infinite series, we give in the following theorem:

**Theorem 6.6** *If* $\sum_{k=0}^{\infty} a_k$ *is convergent then* $\lim_{n\to\infty} a_n = 0$.

**Proof.** Suppose that

$$\sum_{n=0}^{\infty} a_n = S.$$

Then

$$\lim_{n\to\infty} S_n = S \qquad \text{and} \qquad \lim_{n\to\infty} S_{n-1} = S.$$

Since $S_n - S_{n-1} = a_n$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Therefore, $\lim_{n->\infty} a_n = 0$ is the necessary condition for convergence of the infinite series $\sum_{n=0}^{\infty} a_n$.

**Example 6.9** *Show that the infinite series*

$$\sum_{n=0}^{\infty} r^n,$$

*where* $|r| \geq 1$, *is divergent.*

**Solution.** Recall that $\lim_{n\to\infty} r^n = \infty$, when $|r| > 1$ and $\lim_{n\to\infty} r^n = 1$, when $|r| = 1$. Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n \neq 0$$

and the necessary condition for convergence is not satisfied. Consequently, $\sum r^n$ is divergent, when $|r| \geq 1$.

**Example 6.10** *Find the sum of the infinite series* $\sum_{n=0}^{\infty} r^n$, $|r| < 1$.

**Solution.** We have

$$\begin{aligned} S_n &= 1 + r + r^2 + \cdots + r^n \\ rS_n &= \phantom{1+} r + r^2 + \cdots + r^n + r^{n+1}, \end{aligned}$$

and $S_n - rS_n = 1 - r^{n+1}$ which gives

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

Since $|r| < 1$,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1 - r\lim\limits_{n\to\infty} r^n}{1-r} = \frac{1}{1-r}.$$

Hence we have obtained the required result:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1. \tag{6.7}$$

**Example 6.11** *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

*satisfies the necessary condition of convergence, since the term* $a_n = \frac{1}{n} - > 0$, *when* $n- > \infty$. *However, the series diverges to infinity, since the sequence of its partial sums diverges to* $\infty$. *Indeed, we have*

$$\begin{aligned} S_{2n} - S_n &= \frac{1}{n+1} + \ldots + \frac{1}{2n} \geq \frac{1}{n+1} + \frac{1}{n+2} \\ &+ \frac{1}{2n} \geq n\frac{1}{2n} = \frac{1}{2} \end{aligned}$$

*Hence, the sequence* $\{S_{2n} - S_n\}$ *is divergent, since* $S_{2n} - S_n \geq \frac{1}{2}$, *and it does not satisfies the necessary condition of convergence. Therefore, the sequence of partial sums* $\{S_n\}$ *diverges. too.*

### 6.2.1 Absolute Convergence of Infinite Series

Consider an infinite series $\sum\limits_{k=0}^{\infty} |a_k|$. The sequence of its partial sums

$$S_n = |a_o| + |a_1| + |a_2| + \cdots + |a_n| \tag{6.8}$$

is clearly increasing. If $\{S_n\}$ converges, then $\{S_n\}$ must be bounded. Conversely, if $\{S_n\}$ is bounded, then, being monotone increasing, $\{S_n\}$ converges. Therefore the following theorem holds.

Series $\sum\limits_{n=1}^{\infty} a_n$ for which $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent are very important in the theory of series.

A series $\sum\limits_{n=1}^{\infty} a_n$ such that $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent is called absolutely convergent.

For testing convergence of series, we have a few tests, Comparison Test, Root Test,Ratio Test and Alternating Series Test. Firstly, let us state and illustrate the Comparison Test.

### 6.2.2 Comparison Test

. Suppose that

$$0 \le a_k \le b_k, \quad k = 0, 1, 2, \ldots$$

Then,

- if $\sum_{k=0}^{\infty} b_k$ converges, then the series $\sum_{k=0}^{\infty} a_k$ converges.
- if $\sum_{k=0}^{\infty} a_k$ diverges, then the series $\sum_{k=0}^{\infty} b_k$.

**Proof.** Let $S_n$ and $T_n$ denote the $n$-th partial sums:

$$S_n = a_o + a_1 + a_2 + \cdots + a_n,$$

$$T_n = b_o + b_1 + b_2 + \cdots + b_n.$$

Then

$$0 \le S_n \le T_n, \quad n = 1, 2, \ldots \tag{6.9}$$

By the assumption, $\sum_{n=0}^{\infty} b_n$ converges, so $\{T_n\}$ is bounded and implies that $\{S_n\}$ is also bounded. Hence, $\{S_n\}$ is increasing and bounded, by theorem on monotone sequences, $\{S_n\}$ converges. This completes the proof.

**Example 6.12** *Let the series* $\sum_{n=1}^{\infty} a_n,\ a_n \ge 0,\ n = 1, 2, ...$ *be absolutely convergent, that is, the series*

$$\sum_{n=1}^{\infty} |a_n|$$

*converges. Clearly, we have the inequality*

$$0 \le a_n \le |a_n|, \quad n = 1, 2, ...$$

*By the comparison test the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

**Example 6.13** *Consider the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

By the second part of the comparison test, the series diverges for $0 < s \le 1$. Indeed, let

$$b_n = \frac{1}{n^s}, \quad a_n = \frac{1}{n}, \quad n = 1, 2, ...$$

Then, we have

$$0 < a_n = \frac{1}{n} \leq \frac{1}{n^s} = b_n, \quad 0 < s \leq 1, \quad n = 1, 2, ...$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges. too, when $0 < s \leq 1$.
Now, let be $s \geq 2$. By the first part of the comparison test, the series converges for $s > 2$. Indeed, let

$$b_n = \frac{1}{n(n-1)}, \quad a_n = \frac{1}{n^s}, \quad n = 2, 3, ...$$

Then, we have

$$0 < a_n = \frac{1}{n^s} \leq \frac{1}{n(n-1)} = b_n, \quad s \geq 2, \quad n = 2, 3...$$

The series $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ converges (see the example), so the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges. too, when $s \geq 2$.

[1]

### 6.2.3 Cauchy Root Test

. Consider the series

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0, \quad n = 1, 2, ...$$

- If the limit $\lim_{n->\infty} \sqrt[n]{a_n} = a < 1$ exists and it is less then one, then the series converges
- If the limit $\lim_{n->\infty} \sqrt[n]{a_n} = a > 1$ exists, and it is greater than one, then the series diverges
- If the limit $\lim_{n->\infty} \sqrt[n]{a_n} = 1$ exists and it is equal to one, then there is no conclusion.

**Example 6.14** *Consider the series*

$$(i) \quad \sum_{n=1}^{\infty} \frac{n}{5^n}, \qquad (ii) \quad \sum_{n=1}^{\infty} \frac{x^n}{3^n}, \quad x > 0.$$

[1] In the case when $1 < s < 2$ the series converges, but different test is to be used.

$(i)$ We find

$$\lim_{n->\infty} \sqrt[n]{\frac{n}{5^n}} = \frac{\lim_{n->\infty} \sqrt[n]{n}}{\lim_{n->\infty} \sqrt[n]{5^n}} = \frac{1}{5} < 1$$

Since the limit $a = \frac{1}{5} < 1$, by the root test the series converges.

$(ii)$ We find

$$\lim_{n->\infty} \sqrt[n]{\frac{x^n}{3^n}} = \frac{x}{3}$$

Then, by the root test, the series converges for $\frac{x}{3} < 1, \quad 0 \leq x < 3$, and the series diverges $\frac{x}{3} > 1, \quad x > 3$. For $x = 3$ the root test does not apply. However, the series diverges, since for $x = 3$ the necessary condition of convergence is not satisfied.

### 6.2.4 d'Alembert's Ratio Test

. Consider the series

$$\sum_{n=1}^{\infty} a_n, \qquad a_n \neq 0, \quad n = 1, 2, ...$$

Assume that the limit exists

$$\lim_{n->\infty} |\frac{a_{n+1}}{a_n}| = a$$

Then,

- If $a < 1$ the series converges absolutely.
- If $a > 1$ or $a = \infty$ the series diverges.
- If $a = 1$, there is no conclusion.

**Example 6.15** *Use the ration test to investigate convergence of the series*

$$(i) \quad \sum_{n=1}^{\infty} \frac{2^n}{n!}, \qquad\qquad (ii) \quad \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

*We evaluate*

$$\lim_{n->\infty} \frac{a_{n+1}}{an} = \lim_{n->\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n->\infty} \frac{2}{n+1} = 0$$

*By the ration test $a = 0 < 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.*

*For the second series, we evaluate*

$$\lim_{n->\infty} \frac{a_{n+1}}{a_n} = \lim_{n->\infty} \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \lim_{n->\infty} \frac{x}{n+1} = 0$$

*By the ration test* $a = 0 < 1$, *for any real* $x$, *the series* $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ *converges for* $-\infty < x < \infty$.

### 6.2.5 Alternating Series Test

. Suppose that

$$\lim_{n->\infty} a_n = 0 \quad \text{and} \quad 0 < a_{n+1} \leq a_n, \;\; n = 1, 2, ...$$

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n.$$

is convergent.

**Proof.** Consider the partial sums with even subscript $2n$

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + ... + (a_{2n-1} - a_{2n}$$

By the assumption $a_{k+1} \leq a_k, \;\; k = 1, 2, ..., 2n - 1$, the terms $(a_k - a_{k+1}) \leq 0, \;\; k = 1, 2, ..., 2n - 1$. The sequence of partial sums $\{S_{2n}\}$ is increasing.
Now, let us write the partial sum as follows:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - ... - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

Hence, we observe the sequence $\{S_{2n}\}$ is bounded above by $a_1$. So that

$$S_{2n} \leq a_1, \quad n = 1, 2, ...$$

Therefore, by the theorem on monotone sequences, the sequence converges and

$$\lim_{n->\infty} S_{2n} = S$$

Now, let us consider the sequence $\{S_{2n-1}\}$ of partial sums with add subscripts

$$S_{2n-1} = S_{2n} + a_{2n-1}$$

By the assumption $a_n -> 0$ when $n -> \infty$. Therefore, from the above relation the limit of the sequence $\{S_{2n-1}\}$ exists and

$$\lim_{n->\infty} S_{2n-1} = S$$

Thus, the alternating series is convergent and its sum

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = S.$$

**Example 6.16** *Consider the un harmonic series*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n-1}\frac{1}{n}+...=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

We note that it is alternating series with the coefficients

$$0<a_{n+1}=\frac{1}{n+1}<a_n=\frac{1}{n},\quad n=1,2,...$$

and with the limit

$$\lim_{n->\infty}a_n=\lim_{n->\infty}\frac{1}{n}=0$$

By the alternating series test, the un harmonic series is convergent. However, the un harmonic series is not absolutely convergent.

**Example 6.17** *Consider the un harmonic series*

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+...+(-1)^{n-1}\frac{1}{\sqrt{n}}+...=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

We note that it is alternating series with the coefficients

$$0<a_{n+1}=\frac{1}{\sqrt{n+1}}<a_n=\frac{1}{\sqrt{n}},\quad n=1,2,...$$

and with the limit

$$\lim_{n->\infty}a_n=\lim_{n->\infty}\frac{1}{\sqrt{n}}=0$$

By the alternating series test, the series is convergent. However, by the comparison test, comparing with harmonic series, the series is not absolutely convergent.

## 6.3 Exercises. Set 3.

**Question 6.1** *Use L'Hôpital's Rule to evaluate the following limits :*

**(a)** $\lim_{x\to 0}\frac{\sin ax}{x}$ for $a\neq 0$.

**(b)** $\lim_{x\to 0}\frac{\sin 5x}{10x}$.

**(c)** $\lim_{x\to 0}\frac{\cos^2 x-1}{x^2}$.

**(d)** $\lim_{x\to 0}\frac{1-\cos x^2}{\sin^2 x}$

**(e)** $\lim_{h\to 0}\frac{e^h-1}{h}$; **(i)** $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$.

**Question 6.2** *By repeating application of L'Hôpital rules, show that*

$$(\mathbf{a})\quad \lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} = 0.$$

$$(\mathbf{b})\quad \lim_{x\to 1}\frac{x^3 - x^2 - x + 1}{x^3 - 2x^2 + x} = 2.$$

**Improper Integrals**

**Question 6.3** *Evaluate the integrals*

*(a)*

$$\int_{-\infty}^{\infty}\frac{dx}{4+x^2}$$

*(b)*

$$\int_0^{\infty}\frac{x\,dx}{(1+x^2)^2}$$

*(c)*

$$\int_0^1\frac{x\,dx}{\sqrt{1-x}}$$

*(d)*

$$\int_0^1\frac{dx}{\sqrt{1-x^2}}$$

*(e)*

$$\int_0^{\infty} xe^{-x^2}\,dx$$

*(f)*

$$(i)\quad \int_0^{\infty} e^{-x}\sin x\,dx,\qquad (ii)\quad \int_0^{\infty} e^{-x}\cos x\,dx$$

**Question 6.4** *Show the integral*

$$\int_0^{\infty}\frac{dx}{(1+x)^{\alpha}}$$

*is convergent for* $\alpha > 1$ *and divergent for* $\alpha \le 1$.

**Question 6.5** *Find the limit of the sequences, when* $n- > \infty$

$$(a)\quad a_n = \frac{n^2+n+1}{2n(n+1)},\qquad (b)\quad b_n = \sqrt{n}(\sqrt{n+1}-\sqrt{n})$$

$$(c)\quad c_n = (n+1)^5 - n^5,\qquad (d)\quad d_n = \sqrt[n]{(n+1)^2}$$

$$(e)\quad x_n = (1+\frac{2}{n})^n,\qquad (f)\quad y_n = (1-\frac{1}{n^2})^n$$

**Question 6.6** *Let $a > b > 0$ be two given real numbers. Show that the following sequences are monotone and bounded. Find their limits when $n- > \infty$*

$$(i) \quad x_1 = \sqrt{2}, \qquad x_{n+1} = \sqrt{2 + x_n}, \qquad n = 1, 2, ...,$$

$$(i) \quad \text{arithemetic verages} \quad a_1 = \frac{a+b}{2}, \quad a_{n+1} = \frac{a_n + b_n}{2}, \qquad n = 1, 2, ...,$$

$$(ii) \quad \text{geometric verages} \quad b_1 = \sqrt{a\, b}, \quad b_{n+1} = \sqrt{a_n\, b_n} \qquad n = 1, 2, ...,$$

$$(iii) \quad c_n = \frac{3^n}{3^n + 1} \qquad (iv) \quad d_{n+1} = \frac{n^2 + 1}{3n^2 + 1} \quad n = 1, 2, ...,$$

**Question 6.7** *Show that*

$$\lim_{n->\infty} \sqrt[n]{3^n + 9^n} = 9, \qquad (ii) \quad \lim_{n->\infty} \sqrt[n]{\alpha^n + \beta^n} = \max(\alpha, \beta), \quad \alpha > 0,\ \beta > 0.$$

**Question 6.8** *State and use the comparison test to investigate convergence of the series*

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+2)}}, \qquad (b) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+2)}},$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{1}{1+a^n}, \quad a > 0, \qquad (d) \quad \sum_{n=1}^{\infty} \frac{n!}{n^n},$$

**Question 6.9** *State and use the root test to investigate convergence of the series*

$$(a) \quad \sum_{n=1}^{\infty} \frac{n}{3^n}, \qquad (b) \quad 1 + \sum_{n=1}^{\infty} \frac{x^n}{n^n},$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n}, \qquad (d) \quad \sum_{n=1}^{\infty} \frac{x^n}{(\ln\ n)^n},$$

**Question 6.10** *State and use the ratio test to investigate convergence of the series*

$$(a) \quad \sum_{n=1}^{\infty} \frac{2^n}{n!}, \qquad (b) \quad 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}, \qquad (d) \quad \sum_{n=1}^{\infty} n\, x^{n-1}$$

$$(e) \quad \sum_{n=1}^{\infty} \frac{2^n x^n}{n^n}, \qquad (f) \quad \sum_{n=1}^{\infty} \frac{3^n x^n}{n^3}$$

**Question 6.11** *Investigate convergence of the series*

$$(a)\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}, \quad x > 0 \qquad (b)\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n^2}, \quad x > 0,$$

$$(c)\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\ln{(n+1)}}, \qquad (d)\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{\sqrt{n}}$$

$$(e)\quad \sum_{n=1}^{\infty} \frac{3^n x^n}{8^n}, \qquad (f)\quad \sum_{n=1}^{\infty} \frac{4^n x^n}{5^n\ n^2}$$