LECTURE NOTES
in
ELEMENTARY COMPLEX FUNCTIONS
with COMPUTER

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PREFACE

These lecture notes are designed for undergraduate students as a complementary text to complex variables with `Mathematica`. It is assumed that students have basic knowledge in real analysis and computing.

The notes has been used in the course on complex variables given to undergraduate students at the Faculty of Science, University of Botswana. They contain instructions and programs in `Mathematica` as a system for doing mathematics with a computer.
Each chapter ends with a number of questions that can be used for tutorials and tests.

Students are encouraged to learn complex variables by solving tutorial questions with `Mathematica`.

Tadeusz Styš

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Chapter 1

Revision

1.1 Complex Numbers

Every complex number z has the following form:

$$z = x + iy,$$

where

$$x = Re\ z \ \ is\ the\ real\ part\ of\ z$$

$$y = Im\ z \ \ is\ the\ imaginary\ part\ of\ z$$

$$i^2 = -1 \quad is\ the\ imaginary\ unit.$$

In `Mathematica`, real and imaginary parts of a complex number $z = x + iy$ are given by the commands `Re[z]` and `Im[z]`. For example, the output of the commands

```
z=3+4 I ;
Re[z]^2+Im[z]^2
```

is 25.
A complex number $z = x + iy$ can be considered as a point (x, y) on the cartesian plane with the coordinates x and y.

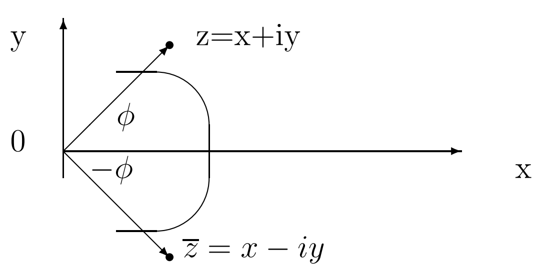

Fig 1.1 Complex Plane

Trigonometric form of z. Also, every complex number z can be written in polar coordinates $(\phi, |z|)$, that is

$$z = |z|(\cos\phi + i\sin\phi),$$

where the modulus $|z| = \sqrt{x^2 + y^2}$ and the argument ϕ is determined by the equalities

$$x = |z|\cos\phi, \qquad y = |z|\sin\phi,$$

for $z \neq 0$.
In `Mathematica`, the module and the principal value of argument of $z = x + iy$ are given by the commands `Abs[z]` and `Arg[z]`. For example, the output of the commands:

```
z=1+I;
 module=Abs[z];
 argument=Arg[z]
```

are: `module=`$\sqrt{2}$ and `argument=`$\frac{\pi}{4}$.
Conjugate complex number. For any complex number z its conjugate is

$$\overline{z} = x - iy.$$

Thus, in the trigonometric form, the conjugate $\overline{z} = |z|(\cos\phi - i\sin\phi)$ has the same modulus as z, i.e.

$$|\overline{z}| = |z|,$$

and the argument of the conjugate is mines $Arg(z)$, i.e.,

$$Arg(\overline{z}) = -Arg(z) = -\phi.$$

One can get the conjugate of $z = x + iy$, by the `Mathematica` command `Conjugate[z]`.
Exponential form of z. Let $z = x + iy$, or in trigonometric form

$$z = |z|(\cos\phi + i\sin\phi).$$

Then, we have the following exponential form of z

$$z = |z|e^{i\phi},$$

where $e^{i\phi} = \cos\phi + i\sin\phi$.
The `Mathematica` function `trigForm` prints the trigonometric form of a complex number z

```
trigForm[z_]:=Print[Abs[z],"(Cos ",Arg[z],"+I Sin ",Arg[z],")"];
```

For example, the command

```
trigForm[1+I]
```

prints the trigonometric form of $z = 1 + i$, as follows:

$$\sqrt{2}(cos\frac{\pi}{4} + I\sin\frac{\pi}{4}).$$

The principal argument of z. Let us note that if ϕ is an argument of z then $\phi + 2k\pi$ is also an argument of z for any integer $k = 0, \pm 1, \pm 2, ...,$
The principal argument of z is the unique one which belongs to the interval $(-\pi, \pi]$, and is denoted by $Arg(z)$. So that

$$-\pi < Arg(z) \leq \pi.$$

Arithmetic operations. We perform four arithmetic operations on the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, according to the following rules
Addition and Subtraction

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2),$$

Multiplication.

$$z_1 * z_2 = (x_1 + iy_1) * (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Division.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2},$$

for $z_2 \neq 0$.
Let us note that multiplying or dividing two complex numbers $z_1 = |z_1|e^{i\phi_1}$ and $z_2 = |z_2|e^{i\phi_2}$, in exponential forms, we find

$$z_1 * z_2 = |z_1|\ |z_2|e^{i(\phi_1+\phi_1)},$$

and

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\phi_1-\phi_2)},$$

for $z_2 \neq 0$.
Power of z. Let us consider z in the exponential form

$$z = |z|e^{i\phi}.$$

Clearly, the power α of z is

$$z^\alpha = |z|^\alpha e^{i\alpha\phi} = |z|^\alpha(\cos\alpha\phi + i\sin\alpha\phi),$$

for any real number α.
In particular, we have De Moivre's formula

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi = e^{i\ n\phi},$$

for any natural n.
In order to convert a complex number from its trigonometric form to the exponential form, we can use the `Mathematica` command `TrigToExp[z]`. For example, the command

```
TrigToExp[Cos[Pi/8]+ I Sin[Pi/8]]
```

gives the exponential form $e^{I\frac{\pi}{8}}$.
In order to covert a complex number from its exponential form to the trigonometric form, we use the command `ExpToTrig[z]`. For example, the command

```
ExpToTrig[Exp[I Pi/8]]
```

gives the trigonometric form $\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$.

1.2 The Root of z

Every complex number $z = x + iy$ which satisfies the equation

$$z^n = a$$

is called n-th root of the complex number $a = a_1 + ia_2$ and denoted by $\sqrt[n]{a}$. The following theorem holds:

Theorem 1.1 *There are exactly n distinct roots of n-th root of a complex number $a \neq 0$. These roots are given by the following formula:*

$$z_k = \sqrt[n]{|a|}(\cos\frac{\phi + 2k\pi}{n} + i\sin\frac{\phi + 2k\pi}{n}), \tag{1.1}$$

for $k = 0, 1, ..., n-1.$, where $\phi = Arg(a)$ and $\sqrt[n]{|a|}$ is the arithmetic root of the real number $|a|$.

Proof. Let us consider the complex numbers

$$z = |z|(\cos\theta + i\sin\theta), \quad a = |a|(\cos\phi + i\sin\phi).$$

Clearly, the equation

$$z^n = a$$

takes the trigonometric form

$$|z|^n(\cos\theta + i\sin\theta)^n = |a|(\cos\phi + i\sin\phi).$$

Hence, by De Moivre's formula

$$|z|^n(\cos n\theta + i\sin n\theta) = |a|(\cos\phi + i\sin\phi).$$

Comparing the modules and arguments, we find

$$|z| = \sqrt[n]{|a|}, \quad n\theta = \phi + 2k\pi, \quad k = 0, \pm, 1, \pm, 2, ...,$$

Thus, all distinct roots of a are

$$z_k = \sqrt[n]{|a|}(\cos\frac{\phi + 2k\pi}{n} + i\sin\frac{\phi + 2k\pi}{n}),$$

for $k = 0, 1, ..., n-1$.

Example 1.1 *Find all roots of the equation*

$$z^n = 1.$$

Let us note that for $a = 1$, we have $|a| = 1$ and $Arg(a) = \phi = 0$. By the formula (1.1), we obtain the following distinct roots:

$$z_k = (\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}),$$

for $k = 0, 1, ..., n-1$.
In the case when $n = 8$, the roots are

$$z_0 = 1, \quad z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_2 = i, \quad z_3 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$
$$z_4 = -1, \quad z_5 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \quad z_6 = -i, \quad z_7 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

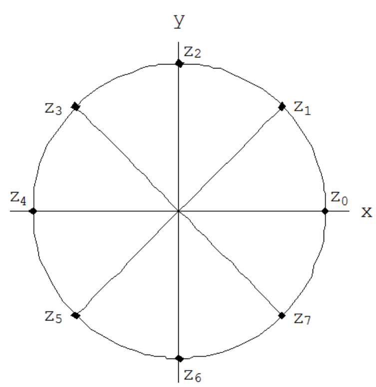

Fig. 1.2 Root $\sqrt[8]{1}$

Let us note that the `Mathematica` command

```
z^(1/n)
```

gives the principal value of the n-th distinct root of z. Also, the command `Sqrt[z]` gives the square root of z.
The module `nroot` gives the list of all n-th roots of a complex number z.

```
nroot[z_,n_]:=Module[{a,al },
a=Abs[z];
al=Arg[z];
Table[a^{(1/n)}(Cos[(al+2 Pi k)/n]+
               I Sin[(al+2Pi k)/n]),{k,0,n-1}]
```

For example, the output of the command

```
nroot[1,4]
```

is the following list of the roots: `1,I,-1,-I`.

1.3 Logarithm of Complex Numbers

Let $a \neq 0$ be a complex number. Every number z which satisfies the equation

$$e^z = a, \; ; \;\; z = x + iy, \tag{1.2}$$

is called *logarithm of* a and denoted by

$$z = ln\ a.$$

The logarithm of $a = 0$ does not exist.
Let us consider a in the exponential form

$$a = |a|e^{i\phi}.$$

Then, the equation (1.2) is

$$e^z = e^{x+iy} = e^x e^{iy} = |a|e^{i\phi}.$$

Hence, we get

$$e^x = |a|, \quad x = ln\ |a|, \quad y = Arg(a) + 2\pi k, \quad k == 0, \pm 1, \pm 2, ...,$$

Thus, there are infinite number of logarithms of a complex number $a \neq 0$ which are given by the formula

$$ln\ a = ln\ |a| + i\ Arg(a) + i\ 2\pi k, \qquad k = 0, \pm 1, \pm 2, ...,$$

However, there is only one principal value of the logarithm

$$ln\ a = ln|a| + i\ Arg(a),$$

which corresponds to the principal argument $Arg(a)$ of a, $(k = 0)$.

Example 1.2 *We compute*

$$ln\ (-1) = ln\ 1 + i\pi + i\ 2\pi k = i(2k+1)\pi, \qquad k = 0, \pm 1, \pm 2, ...,$$

and the principal value $\ln(-1) = i\pi$.

The command `Log[z]` in `Mathematica` gives the principal value of the logarithm of z. For example, the output of the command `Log[-1]` is Iπ.

1.4 Exercises

Question 1.1 *Evaluate*

$$(i) \quad \sqrt[8]{-1}, \qquad (ii) \quad \sqrt[4]{1+i}.$$

Question 1.2 *Use* `Mathematica` *to evaluate*

$$(i)\quad \frac{z^2+2z+1}{z^4+2z^2+1}, \qquad (ii)\quad (\overline{z^2+z+1})\sqrt[4]{z},$$

for $z = 1 + i$.

Question 1.3 *Let* $a = a_1 + ia_2$ *and* $b = b_1 + ib_2$, *be two complex numbers different from zero. For which values of their arguments the product* $a\,b$ *and the quotient* $\frac{a}{b}$ *are real numbers.*

Question 1.4 *Prove that*

1. (a)
$$\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2,$$
(b)
$$\overline{z_1\ z_2} = \overline{z_1}\ \overline{z_2},$$
(c)
$$\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}, \qquad z_2 \neq 0,$$
(d)
$$|z_1\ z_2| = |z_1|\ |z_2|,$$
(e)
$$(i)\quad |Re\ z| \leq |z|, \qquad (ii)\quad |Im\ z| \leq |z|$$
(f)
$$|z_1 \pm z_2| \leq |z_1| + |z_2|,$$
(g)
$$|z_1 \pm z_2|^2 = |z_1|^2 \pm 2Re(z_1\ \overline{z}_2) + |z_2|^2.$$
(h) *Check the relations (a), (b) and (d) in* `Mathematica`.

Question 1.5 *Assume the* $z_k \neq 1$ *is an n-th root of one. Show that*

$$1 + z_k + z_k^2 + \ldots + z_k^{n-1} = 0.$$

Question 1.6 *Show that*

$$\left|\frac{1-z}{\overline{z}-1}\right| = 1, \qquad z \neq 1.$$

Question 1.7 *Show that*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|.$$

for complex numbers $z_1, z_2, \ldots, z_n$.

Question 1.8 *Prove that*

1. (a)

$$\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\sin\frac{6\pi}{n}+\cdots+\sin\frac{2(n-1)\pi}{n}=0,$$

(b)

$$\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\cdots+\cos\frac{2(n-1)\pi}{n}=-1.$$

for any even $n=2,4,...,$
Hint: Solve the equation $z^n-1=0$.

Question 1.9 *Prove the following formula:*

1. (a)

$$\cos n\phi=\cos^n\phi-\binom{n}{2}\cos^{n-2}\phi\ \sin^2\phi+\binom{n}{4}\cos^{n-4}\phi\sin^4\phi-...,$$

(b)

$$\sin n\phi=\binom{n}{1}\cos^{n-1}\phi\ \sin\phi-\binom{n}{3}\cos^{n-3}\phi\sin^3\phi+\binom{n}{5}\cos^{n-5}\phi\sin^5\phi-...,$$

Question 1.10 *Sketch the following sets*

1. (a)

$$D=\{z\in Z:\ \ |z-i|<|z-1|\}.$$

(b)

$$D=\{z\in Z:\ \ |z|^2>z+\overline{z}\}.$$

Question 1.11 *Show that*

$$\frac{1}{2}|a+b|\leq\max\{|a|,|b|\},$$

for every complex numbers a *and* b.

Question 1.12 *Show that*

$$|\frac{z-a}{\overline{a}z-1}|=1,$$

for every $|z|=1$ *and* $z\neq a$.

Question 1.13 *Let* $z=re^{i\theta}$ *and* $w=Re^{i\varphi}$, *where* $0<r<R$. *Show that*

$$Re\left(\frac{w+z}{w-z}\right)=\frac{|w|^2-|z|^2}{|w-z|^2}=\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}.$$

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Chapter 2

Sets on the Complex Plane

2.1 Examples of Sets

1. (a) **Line Segment.** For given complex numbers $a = a_1 + ia_2$ and $b = b_1 + ib_2$, the line segment with the end points a and b is the following set:

$$[a, b] = \{z(t) = (1 - t)a + t\, b : \ \ 0 \le t \le 1\},$$

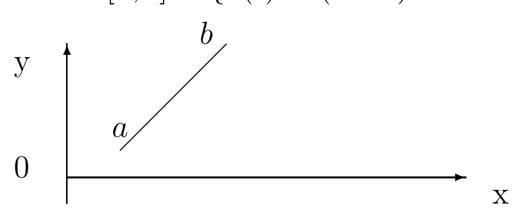

Fig 2.1 Line Segment

(b) **Circle.** The circle $C(z_0, r)$ with the radius $r > 0$ and the center at the point z_0 is the set of the points z which satisfy the equation

$$|z - z_0| = r.$$

Also, the same circle has the following trigonometric equation:

$$z = z_0 + r(\cos\phi + i\sin\phi), \qquad -\pi < \phi \le \pi,$$

or exponential equation

$$z = z_0 + re^{i\phi}, \qquad -\pi < \phi \le \pi.$$

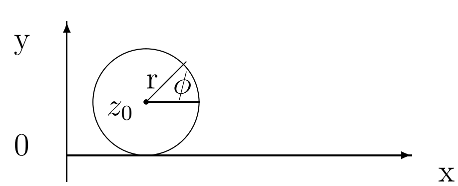

Fig 2.2 Circle

(c) **Disc**. The disc $D(z_0, r)$ with the center at z_0 and radius $r > 0$ is determined by the following inequality

$$|z - z_0| < r.$$

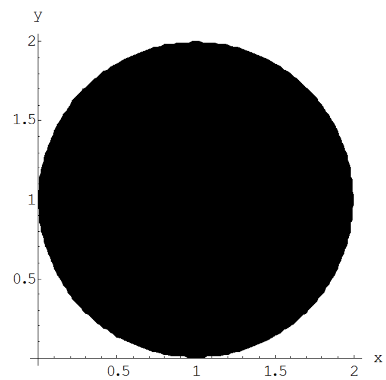

Fig.2.3 Disc(z_0, r)

(d) **Annulus.** The annulus $A(z_0, r_1, r_2)$ with the center at z_0 and the radii $0 < r_1 < r_2$ is the set of all points z which satisfy the following inequality:

$$r_1 < |z - z_0| < r_2,$$

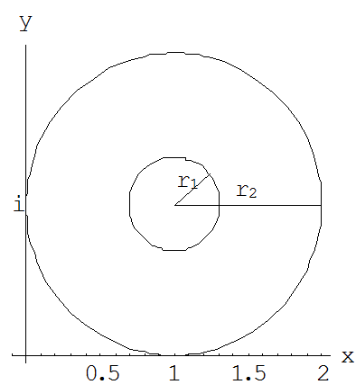

Fig.2.4 Annulus(z_0, r_1, r_2)

(e) **Strip**. The strip of the width $2r$ in the direction of x axis or of y axis is the set of points which satisfy the following inequality, either

$$-r < Im\ z < r,$$

or

$$-r < Re\ z < r,$$

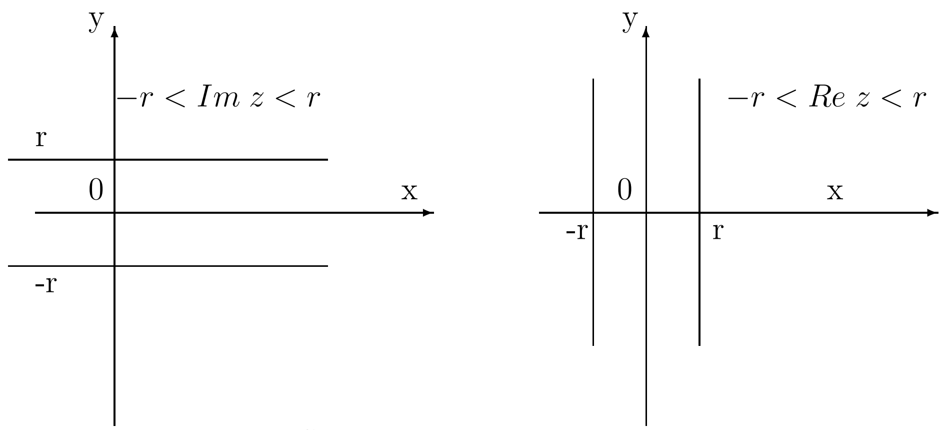

Fig 2.5 Strip

(f) **Sector**. The sector with the angle between α and β is the set of all points z which satisfy the following inequality:

$$\alpha < Arg(z) < \beta,$$

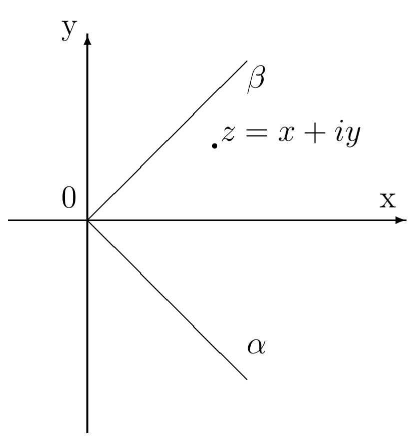

Fig 2.6 Sector

Neighborhood. The $\epsilon-$ neighborhood of a complex number z_0 is the disc

$$N_\epsilon(z_0, z) = \{z \in Z : \ |z - z_0| < \epsilon, \} \quad \epsilon > 0,$$

where Z is the complex plane.
Interior, exterior and boundary complex numbers of a set D. A complex number z_0 is an interior number of the set D, whenever, there is some neighborhood $N(z_0, z)$ of z_0 which is included in D, that is $N(z_0, z) \subset D$. The point z_0 is an exterior complex number of the set D if there is a neighborhood $N(z_0, z)$ which contains no numbers of D. If z_0 is neither of these, it is then a boundary point of D. Thus, z_0 is a boundary number of D if every neighborhood of z_0 contains both interior and exterior numbers of D.
Open Set. A set D of complex numbers is open if it consists only of interior numbers, so that, every number $z_0 \in D$ belongs to D together with its some neighborhood.
Closed Set. A set D of complex numbers is closed if D contains all its interior and boundary numbers.
Let us observe that some sets can be neither open nor closed. For example, the set

$$D = \{z \in Z : \ 0 < |z| \leq 1, \}$$

is neither open nor closed.
Connected Set. An open set D is connected if each pair of numbers $z_1, z_2 \in D$ can be joined by a polygonal path consisting of a finite number of line segments joined end to end which entirely lie in D.
Bounded Set. A set D is bounded if there is a disc $|z| \leq R < \infty$ which contains the set D, otherwise D is an unbounded set.
Domain. An open set D which is connected is called domain.

2.2 Curves on Complex Plane

Let $x(t)$ and $y(t)$ be real continuous functions given for $t_1 \leq t \leq t_2$. Then the parametric equation

$$C: \qquad z(t) = x(t) + iy(t), \qquad t_1 \leq t \leq t_2, \tag{2.1}$$

defines a continuous curve on complex plane joining end points $a = z(t_1)$ and $b = z(t_2)$. If the end points coincide, that is, $a = z(t_1) = z(t_2) = b$, then the curve is said to be closed.
Simple Closed Curve. A continuous closed curve which does not intersects itself is called *simple closed curve.*
Arc. let us assume that $x(t)$ and $y(t)$ are continuously differentiable real functions in the interval $[t_1, t_2]$. Then, the curve C given by the equation (2.1) which does not intersect itself is called *smooth curve* or *arc.*
Contour. A curve which is composed of a finite number of arcs is called *Contour.*

Example 2.1 *The parametric equation of an ellipse on complex plane*

$$z(t) = r_1 \cos\ t + i\, r_2\ \sin\ t, \quad r_2 \geq r_1 > 0, \quad -\pi < t \leq \pi.$$

represents a closed arc.

Example 2.2 *Let*

$$z(t) = \begin{cases} (1+i)t, & 0 \le t < 1, \\ (3-i) + (i-1)t, & 1 \le t \le 2. \end{cases}$$

This equation defines a contour consisting of two segments.

Length of a contour. The length d of a contour $C: \ z(t) = x(t) + iy(t), \ \alpha \le t \le \beta$, open or closed, is given by the formula

$$d = \int_{\alpha}^{\beta} |z^{'}(t)| dt.$$

Indeed, the parametric equations of the contour C on the cartesian plane are

$$x = x(t), \qquad y = y(t), \qquad \alpha \le t \le \beta.$$

As we know, from Mathematical Analysis, the length

$$d = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_{\alpha}^{\beta} |z^{'}(t)| dt.$$

2.3 Exercises

Question 2.1 *Sketch the following sets*

1. (a)
$$D = \{z \in Z: \ |3z - 2 + i| \le 1\},$$
   (b)
$$D = \{z \in Z: \ |z - 4| \ge |z|\},$$
   (c)
$$D = \{z \in Z: \ |Re\ z| < |z|\},$$
   (d)
$$D = \{z \in Z: \ Im\ z^2 > 1\},$$
   (e)
$$D = \{z \in Z: \ |z + \frac{1}{2}| < |z + 1|\}.$$

Question 2.2 *Write the equation of an ellipse, hyperbola, parabola in complex form.*

Question 2.3 *Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.*

Question 2.4 *Represent graphically the set of values of z for which*

1. (a)
$$|\frac{z-3}{z+3}| = 2,$$
(b)
$$|\frac{z-3}{z+3}| < 2,$$
(c)
$$Re\ z^2 > 1.$$

Question 2.5 *Describe and graph the locus represented by each of the following equations*

1. (a) $|z+2i| + |z-2i| = 6$,
   (b) $|z-3| - |z+3| = 4$,
   (c) $z(\overline{z}+2) = 3$.

Question 2.6 *Find the equation of a line passing through the points $z_1 = 1+i$ and $z_2 = 2-3i$.*

Question 2.7 *Show that the equation*

$$|z-4i| + |z+4i| = 10$$

represents an ellipse. Find the equation of this ellipse in the cartesian coordinates x and y and polar coordinates (r, ϕ). Plot the graph of the ellipse with `Mathematica`.

Question 2.8 *Show that the equation*

$$z^2 + \overline{z}^2 = 2$$

represents a hyperbola. Find the equation of the hyperbola in the cartesian coordinates x and y and polar coordinates (r, ϕ). Plot the graph of the hyperbola with `Mathematica`.

Question 2.9 *Find an equation of the circle passing through the points $1-i$ and $1+i$. Plot the circle with* `Mathematica`.

Question 2.10 *Show that the locus of z such that*

$$|z-a||z+a| = a^2, \qquad a > 0,$$

is a lemniscate. Write the equation of the lemniscate in polar coordinates. Plot the graph of the lemniscate with `Mathematica`.

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Chapter 3

Elementary Functions of a Complex Variable

3.1 Definition

Let D be a set of complex numbers. A function f defined on D is a rule that assigns to each $z \in D$ a complex number w. The complex number w is called the value of the function f at the number z, so that, we note

$$w = f(z), \quad z \in D \quad \text{or} \quad f: \, z \in D \; \rightarrow \; w \in D^{'}.$$

The set D is called the domain of the function f, and the set $D^{'}$ of all values of $f(z)$ is called the image of the set D, that is $D^{'} = f(D)$.

3.2 Linear Function.

Consider the linear function

$$f(z) = az + b, \qquad a \neq 0, \quad z \in Z,$$

where the constant coefficients $a = a_1 + ia_2$ and $b = b_1 + ib_2$.
Clearly, the domain of the linear function f is whole complex plane, and the set of all values of $f(z)$ is also the whole complex plane. Thus, $f(z)$ maps complex plane onto itself. Let us note that the linear function $f(z) = az + b$, $a \neq 0$, is one to one mapping. Indeed, to show this, we observe that

$$f(z_1) = f(z_2)$$

if and only if $z_1 = z_2$. Since, the equality

$$az_1 + b = az_2 + b$$

implies $z_1 = z_2$ if $a \neq 0$.
Translation. The mapping

$$w = z + b, \qquad z \in Z,$$

is called translation.
For example, the line $Re\ z = 1$ on $Z-$plane is translated on the line $Re\ w = 2$ on $W-$plane, by the translation $w = z + 1 + i$.

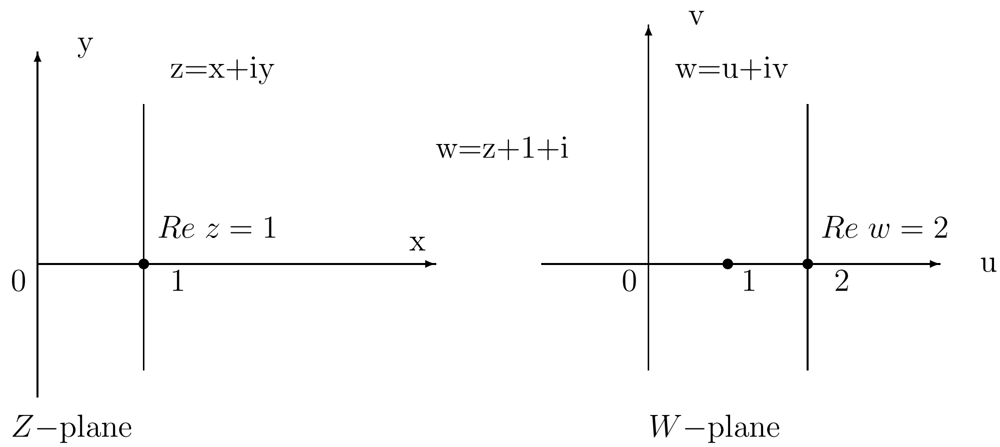

Fig 3.1. Translation $w = z + 1 + i$.

Rotation. The mapping

$$w = az, \qquad |a| = 1, \quad z \in Z,$$

of the $Z-$plane onto $W-$plane is the rotation around the origin by the angle $\alpha = Arg(a)$.
Indeed, we have the following exponential forms

$$a = e^{i\alpha}, \qquad z = |z|e^{i\theta},$$

Hence, we obtain

$$w = |z|e^{i(\alpha+\theta)}.$$

For example, the rotation

$$w = iz, \qquad z \in Z,$$

transforms the line $Re\ z = 1$ on $Z-$plane onto line $Im\ w = 1$ on $W-$plane. Since, we have

$$z = 1 + iy, \quad a = i, \quad \alpha = Arg(i) = \frac{\pi}{2}$$

and

$$w = u + iv = |i|\ |z|e^{i(\theta+\frac{\pi}{2})} = \sqrt{1+y^2}(\cos(\theta + \frac{\pi}{2}) + i\sin(\theta + \frac{\pi}{2})) = -y + i.$$

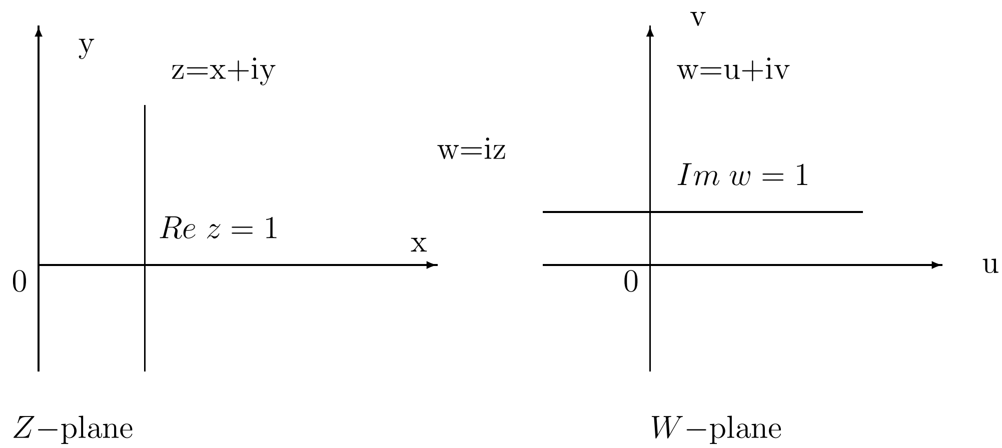

Fig. 3.2 Rotation $w = iz$.

In general, the linear mapping

$$w = az + b, \qquad a \neq 0,$$

is a composition of the rotation

$$s = az, \quad a \neq 0, \quad z \in Z,$$

and the translation

$$t = s + b, \qquad s \in Z,$$

3.3 The Power Function z^n

Let us consider the power function

$$w = z^n, \qquad z \in D = \{z \in Z : \; -\frac{\pi}{n} < Arg(z) \leq \frac{\pi}{n}\}.$$

for natural $n = 1, 2, ...,$
This function maps a sector D onto whole $W-$plane. Indeed, let us write the power function in the following exponential form

$$w = |z|^n e^{n\phi}, \qquad \phi = Arg(z),$$

Clearly, if $z \in D$, that is $-\frac{\pi}{n} \leq Arg(z) \leq \frac{\pi}{n}$, then $-\pi \leq Arg(w) \leq \pi$, and therefore $w \in W$.

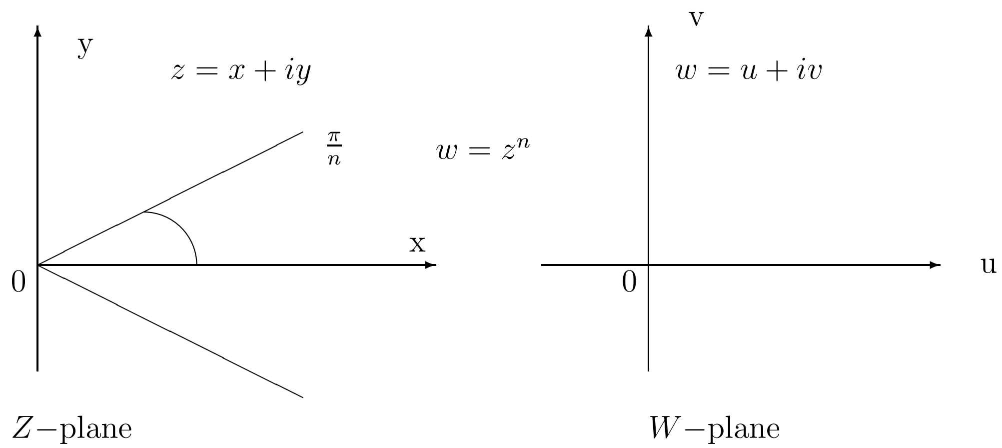

Fig. 3.3 Power Function $w = z^n$.

Let us note that if z moves throughout $Z-$plane than w reaches n times each point of $W-$plane. So that, the function is not an one to one mapping. However, the power function is one to one mapping of the sector

$$D_k = \{z \in Z : \frac{(2k-1)\pi}{n} < Arg(z) \leq \frac{(2k+1)\pi}{n}\}, \quad k = 0, 1, ..., n-1.$$

onto whole $W-$plane.
In `Mathematica`, we compute $(x + i\ y)^n$, by the command

```
(x+I y)^n;
```

3.4 The $n - th$ Root Function

The $n - th$ root function

$$w = \sqrt[n]{z}, \qquad z \in Z,$$

has the following exponential form

$$w = \sqrt[n]{|z|}e^{i\frac{\phi+2\pi k}{n}}, \qquad \phi = Arg(z), \quad k = 0, 1, ..., n-1.$$

Let us note that the n-th root function possesses n different branches for $k = 0, 1, ..., n-1$. In the case when $k = 0$, the function

$$w = \sqrt[n]{|z|}e^{i\frac{\phi}{n}}, \qquad \phi = Arg(z),$$

is called *Principal Branch* of $n - th$ root function.
This function maps whole $Z-$plane onto one of the sectors

$$D_k = \{z \in Z : \frac{(2k-1)\pi}{n} \leq Arg(w) < \frac{(2k+1)\pi}{n}\}, \quad k = 0, 1, ..., n-1.$$

On the figure, we present the graph of the sector D_0 under principal branch of the $n-th$ root function.

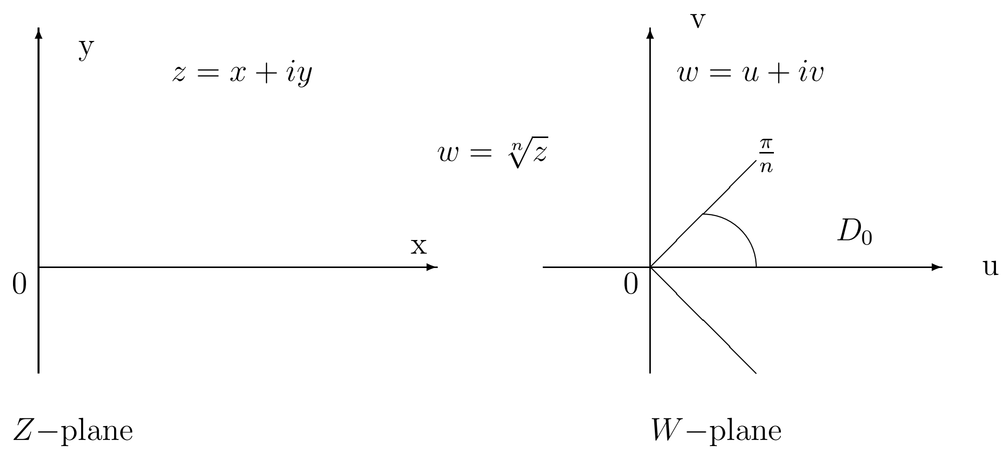

Fig. 3.4 Root Function $w = \sqrt[n]{z}$.

3.5 The Exponential Function $w = e^z$

Let us prove first the following theorem:

Theorem 3.1 *The equation*

$$e^z = 1$$

holds if and only if $z = 2\,k\pi\, i, \quad k = 0, \pm 1, \pm 2, ...,.$

Proof. For $z = x + iy$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = 1.$$

Hence

$$e^x \cos y = 1 \qquad \text{and} \qquad e^x \sin y = 0.$$

So that

$$\sin y = 0, \qquad \text{for} \qquad y_k = k\pi, \quad k = 0, \pm 1, \pm 2, ...,$$

k must be an even integer, since $e^x \cos y_k < 0$ for odd k. Therefore, the equality

$$e^x \cos y = 1$$

holds for $x = 0$ and $y = 2\,\pi\,k$, and $e^z = 1$ if and only if $z = 2\,\pi k\, i$, for any integer k.
From the theorem, it follows that $w = f(z) = e^z$ is a periodic function with the period $\omega = 2\,\pi\, i$. Indeed, we have

$$f(z + 2\pi i) = e^{z+2\pi\, i} = e^z\, e^{2\,\pi\, i} = e^z = f(z).$$

The exponential function $w = f(z) = e^z$ is one to one mapping of the strip

$$D = \{z = x + iy \in Z : \ -\pi < y \leq \pi\}$$

onto whole $W-$plane. Indeed, we have

$$w = u + iv = e^z = e^x(\cos y + i \sin y).$$

So that $-\infty < u = e^x \cos y < \infty$ and $-\infty < v = e^x \sin y < \infty$ when $-\infty < x < \infty$ and $-\pi < y \leq \pi$. Also, we note that $f(z_1) = f(z_2)$ if and only if $z_1 = z_2$ when $z_1, z_2 \in D$. This means that $f(z) = e^z$ maps one to one the strip D to whole $W-$plane.
Clearly, $f(z) = e^z$ is a periodic function if it is considered on whole $Z-$plane, since the function maps every strip

$$D_k = \{z = x + iy \in Z : \ (2k-1)\pi < y \leq (2k+1)\pi, \quad k = 0, \pm 1, \pm 2, ...,$$

onto whole $W-$plane.

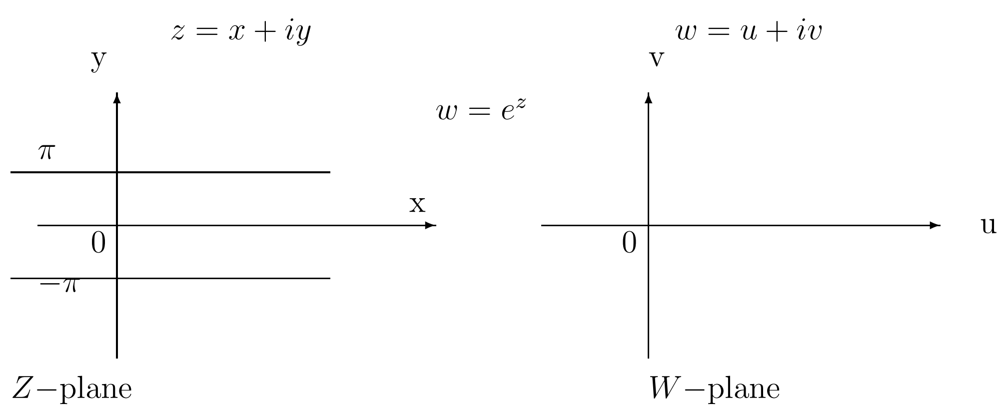

Fig. 3.5 Exponential Function $w = e^z$

3.6 The Logarithmic Function $w = ln\ z$.

As we know, the exponential function maps one to one every strip

$$D_k = \{z = x + iy \in Z : \ (2k-1)\pi < y \leq (2k+1)\pi, \quad k = 0, \pm 1, \pm 2, ...,$$

onto whole $W-$plane, so that, the inverse function exists and it maps $W-$plane (without $z = 0$) onto a strip $D_k, \ k = 0, \pm 1, \pm 2, ...,$. This inverse function is called logarithmic function and is given by the following formula:

$$\ln z = \ln |z| + i(Arg(z) + 2\pi k), \qquad z \neq 0, \ \ k = 0, \pm 1, \pm 2, ...,$$

where $Arg(z)$ is the principal value of the argument of z. Let us note that logarithmic function possesses infinite number of branches. Therefore, $w = ln\ z$ is a multivalued function if it is considered on the whole complex plane. The branch which corresponds to $k = 0$ is called *Principal Branch.* Thus, the principal branch is given by the following formula:

$$\ln z = \ln|z| + iArg(z), \qquad z \neq 0.$$

Example 3.1 *Let us consider the principal branch of the logarithmic function*

$$\ln z = \ln|z| + i\phi, \qquad \phi = Arg(z),$$

Show that the principal branch maps

1. *(a) circles with center at the origin on $Z-$plane onto segments parallel to v axis on $W-$plane,*

   (b) lines on rays emanating from the origin on $Z-$plane onto lines parallel to the u axis on $W-$plane.

   (c) the whole $Z-$plane onto a strip of width 2π on $W-$plane.

Solution.

1. (a) Let $|z| = r$ be a circle on $Z-$plane. The principal branch of the logarithmic function

$$\ln z = \ln|z| + i\phi, \qquad \phi = Arg(z), \qquad -\pi < \phi \leq \pi,$$

maps such a circle onto the segment

$$w = \ln r + i\phi, \qquad -\pi < \phi \leq \pi,$$

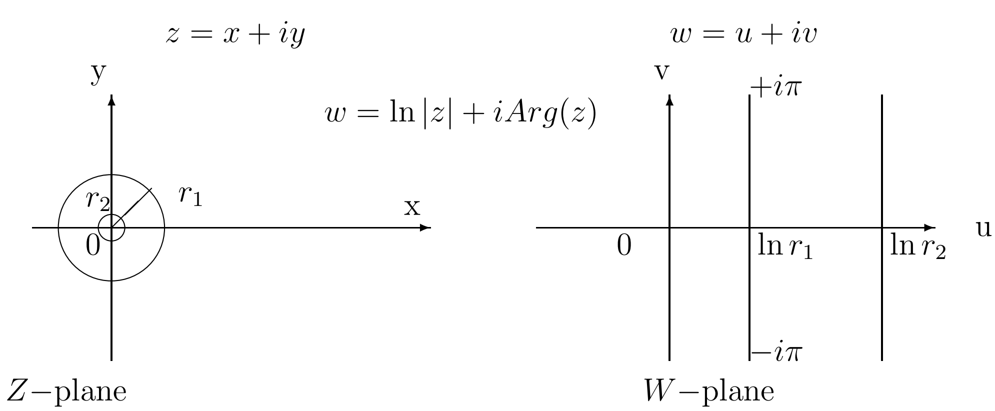

Fig. 3.6 Logarithmic Function $w = \ln|z| + iArg(z)$

(b) A ray from the origin has the following equation

$$Arg(z) = C = constant.$$

Applying the principal branch of the logarithmic function to z with constant argument, we obtain the line parallel to u axis

$$w = u + iv = \ln|z| + iC.$$

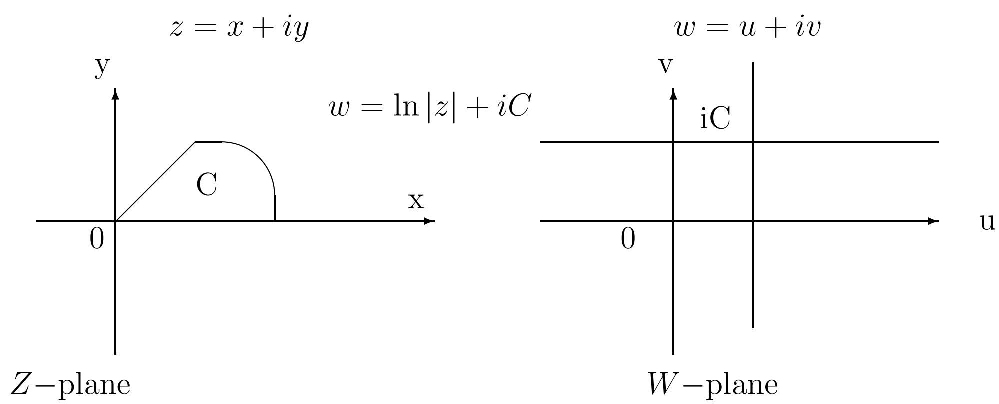

Fig. 3.7 Logarithmic Function $w = \ln|z| + iArg(z)$

(c) The whole $Z-$plane is mapped onto the strip

$$w = u + iv = \ln|z| + iArg(z), \qquad z \in Z, \quad -\pi < Arg(z) \le \pi,$$

since, we have

$$-\infty < u = \ln|z| < \infty, \quad -\pi < v = Arg(z) \le \pi.$$

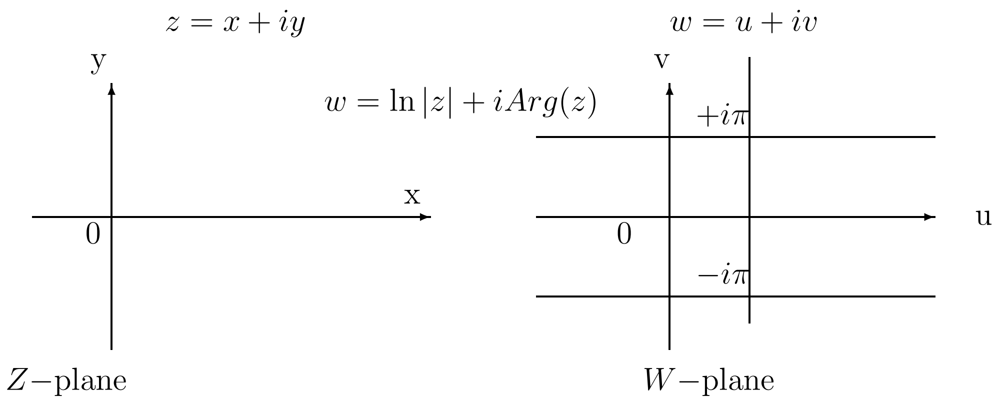

Fig. 3.8 Logarithmic Function $w = \ln|z| + iArg(z)$

3.7 The Trigonometric Functions

The trigonometric functions are related with the exponential function by the following formulas:

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x,$$

from which

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad -\infty < x < \infty.$$

We define, in the same way, *sine* and *cosine* functions of a complex variable z, so that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad z \in Z.$$

For the trigonometric functions *tangent* and *cotangent*, we have formulas

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \quad z \neq (2k-1)\frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, ...,$$
$$\cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}, \quad z \neq k\pi, \quad k = 0, \pm 1, \pm 2, ...,$$

Let us note some of the properties of trigonometric functions which are known in a real variable also hold in a complex variable. For example, we have

$$\sin^2 z + \cos^2 z = 1,$$

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$$

$$\tan(-z) = -\tan z,$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2, \quad z \in Z,$$

However, modulus of $\sin z$ or $\cos z$ can exceed one. Indeed, we have

$$\sin 2i = |\frac{e^{-2} - e^2}{2}| > 3,$$

for $z = 2i$.

3.8 The Hyperbolic Functions.

The hyperbolic functions *sinhz* and *coshz* of a complex variable are given by the formulas

$$sinhz = \frac{e^z - e^{-z}}{2}, \qquad coshz = \frac{e^z + e^{-z}}{2}, \quad z \in Z.$$

These functions satisfy the following identities

$$cosh^2 z - sinh^2 z = 1,$$

$$sinh(-z) = -sinhz, \quad cosh(-z) = coshz,$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

for all $z \in Z$.

3.9 The Function $w = \frac{1}{z}$

The function

$$w = \frac{1}{z}, \qquad z \neq 0,$$

is one to one mapping of non-zero numbers on $Z-$plane onto non-zero numbers on $W-$plane, since

$$\frac{1}{z_1} = \frac{1}{z_2}$$

if and only if $z_1 = z_2$.
The exponential form of this function is

$$w = \frac{1}{|z|}e^{-i\theta}, \qquad -\pi < \theta \leq \pi,$$

for $z = |z|e^{i\theta}$.
Clearly, the function maps a circle

$$|z| = r, \qquad r > 0$$

onto a circle

$$|w| = \frac{1}{r}.$$

Also, under this mapping, the image of the disc

$$0 < |z| < r,$$

is the region

$$|w| > \frac{1}{r},$$

outside of the disc on $W-$plane.

Example 3.2 *Find the image of the line*

$$Re\ z = \alpha \neq 0,$$

under the mapping

$$w = u + iv = \frac{1}{z}, \qquad z = x + iy.$$

Sketch the graph.

Solution. Let us note that

$$w = u + iv = \frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2},$$

Hence, we have

$$u = \frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2}.$$

and

$$u^2 + v^2 = \frac{1}{x^2 + y^2} = \frac{u}{\alpha}, \quad \text{for} \quad Re\ z = x = \alpha.$$

By simple modification, we find

$$u^2 - \frac{u}{\alpha} + \frac{1}{4\alpha^2} + v^2 = \frac{1}{4\alpha^2},$$

and

$$(u - \frac{1}{2\alpha})^2 + v^2 = (\frac{1}{2\alpha})^2.$$

The above equation represents the circle on $W-$plane with the center at $w_0 = \frac{1}{2\alpha}$ and the radius $r = \frac{1}{2\alpha}$. So that, the function maps the line $Re\ \alpha \neq 0$ onto the circle $|w - w_0| = r$.

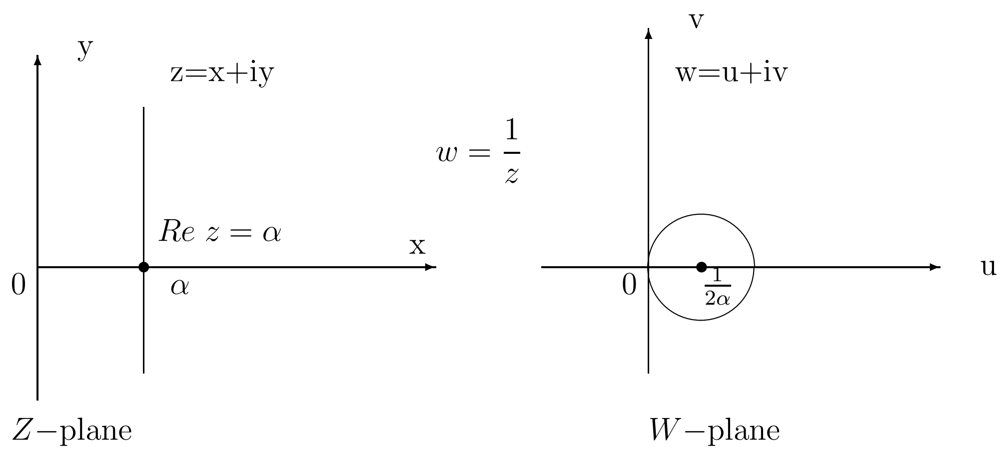

Fig. 3.9 The Function $w = \frac{1}{z}$

3.10 The Linear Fractional Transformation.

Let us consider the linear fractional mapping

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad c \neq 0.$$

This mapping has the following equivalent form

$$w = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz + d}, \quad ad - bc \neq 0, \quad c \neq 0. \tag{3.1}$$

The linear fractional transformation is one to one mapping of complex plane onto itself. Indeed, the inequality

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d},$$

holds if and only if $z_1 = z_2$. Since then

$$(az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d),$$

and

$$(ad - bc)(z_1 - z_2) = 0.$$

Hence, we have $z_1 = z_2$.
The inverse to a linear fractional function is also linear fractional function, since we have

$$z = \frac{-dw + b}{cw - a}, \qquad w \in W.$$

A linear fractional function is the composition of a linear function and the function $w = \frac{1}{z}$.
Indeed, by the formula (3.1), we have

$$w = As + B, \quad A = \frac{bc - ad}{c}, \quad B = \frac{a}{c}, \quad s = \frac{1}{t}, \quad t = cz + d.$$

Example 3.3 *Show that the equation*

$$|\frac{z-p}{z-q}| = \alpha,$$

represents a circle for every $\alpha > 0$, $\alpha \neq 1$ *and* $p \neq q$. *Find the center and the radius of the circle.*

Solution. By the formula

$$|a - b|^2 = |a|^2 + |b|^2 - 2Re\,\overline{(}ab),$$

we have

$$|z - p|^2 = |z|^2 + |p|^2 - 2Re\,\overline{p}z = \alpha^2(|z|^2| + |q|^2 - 2Re\,\overline{q}z) = \alpha^2|z - q|^2.$$

After simple operations, we arrive at the following equation

$$|z|^2 - \frac{2Re\,(\overline{p} - \alpha^2\overline{q})z}{1 - \alpha^2} = \frac{-|p|^2 + \alpha^2|q|^2}{1 - \alpha^2}.$$

Adding to both sides the term $|\frac{p - \alpha^2 q}{1 - \alpha^2}|^2$, we obtain the equation

$$|z - \frac{(p - \alpha^2 q)}{1 - \alpha^2}|^2 = \frac{\alpha^2|p - q|^2}{(1 - \alpha^2)^2}.$$

of the circle with the center at

$$z_0 = \frac{p - \alpha^2 q}{1 - \alpha^2},$$

and the radius

$$r = \alpha\frac{|p - q|}{|1 - \alpha^2|}.$$

Example 3.4 *Consider the linear fractional mapping*

$$w = \frac{z - z_0}{z - \overline{z}_0},$$

where z_0 is a fixed point on the upper half of $Z-$plane, i.e., $Im\ z_0 > 0$. Show that the function maps one to one the upper half of $Z-$plane onto unit disc $|w| < 1$, on $W-$plane. Also, show that every point of x axis is mapped onto unit circle $|w| = 1$.

Solution.

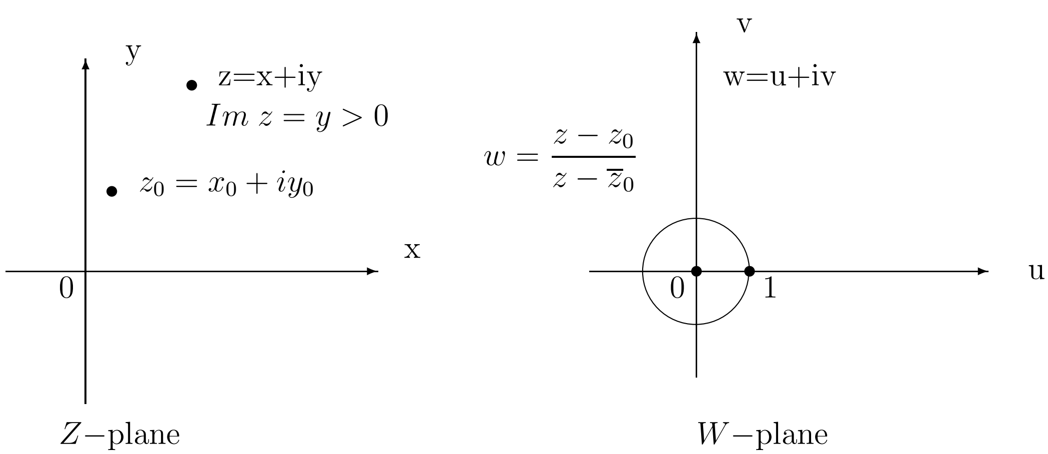

Fig. 3.10 The Function $w = \dfrac{z - z_0}{z - \overline{z}_0}$

Let us note that

$$|w|^2 = \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2} \leq 1,$$

for $Im\ z = y \geq 0$ and $Im\ z_0 = y_0 > 0$.
Clearly, the equality $|w|^2 = 1$ holds if and only if $y = 0$, so that, the x axis $(Im\ z = y = 0)$ is mapped on the circle $|w| = 1$.
To show that the function is one to one mapping, we observe that the equality

$$\frac{z_1 - z_0}{z_1 - \overline{z}_0} = \frac{z_2 - z_0}{z_2 - \overline{z}_0},$$

is equivalent to the following equality

$$(z_1 - z_2)(\overline{z}_0 - z_0) = 0.$$

Hence, for $Imz_0 > 0$, we get $z_1 = z_2$. This means that the linear fractional function is one to one mapping of the upper half of complex plane onto disc $|w| \leq 1$.

3.11 Exercises

Question 3.1 *Find the image of the circle*

$$|z-1|=2$$

under the linear mapping

$$w=(1+i)z+1-i.$$

Write the image in polar coordinates and plot it with `Mathematica`*.*

Question 3.2 *.*

1. (a) *Let $f(z)=z^2$. Evaluate $f(-2+i)$ and $f(1-3i)$*
   (b) *Show that the line joining the numbers $z_1=-2+i$ and $z_2=1-i$ on $Z-$plane is mapped into a curve on $W-$plane joining the numbers w_1 and w_2. Find the equation of the curve in polar coordinates and plot it with* `Mathematica`*.*

Question 3.3 *Find the image of the hyperbola*

$$(i)\quad x^2-y^2=1, \qquad (ii)\quad xy=2.$$

under the mapping $w=z^2$. Plot the image with `Mathematica`*.*

Question 3.4 *Find the image of the sector*

$$0<Arg(z)\leq\frac{\pi}{8}$$

under the mapping

$$w=z^4.$$

Sketch the graph.

Question 3.5 *.*

1. (a) *List all branches of the function*

$$f(z)=\sqrt[3]{z}.$$

   (b) *Find the image of the region*

$$D=\{z\in Z:\ Re\ z\geq 0,\ \ Im\ z\geq 0\}$$

   under the principal branch of $f(z)$. Sketch the graph.

Question 3.6 *Find the image of the line segment*

$$S=\{z\in Z:\ Re\ z=0,\ \ \text{and}\quad -\pi<Im\ z\leq\pi\},$$

under the mapping $w=e^z$. Write the image in polar coordinates and plot it with `Mathematica`*.*

Question 3.7 .

1. (a) *Show that the function*

$$w = f(z) = e^{z^2},$$

maps the lines $y = x$ *and* $y = -x$ *onto unit circle* $|w| = 1$.

(b) *Show further that* $f(z)$ *maps each of the two pieces of the region*

$$D = \{z = x + iy \in Z : \ x^2 > y^2\},$$

onto the set

$$\Omega = \{w = u + iv \in W : \ |w| > 1\}.$$

Question 3.8 . *Solve the following equations:*

1. (a)

$$(i) \quad \ln\, z = \frac{i\pi}{6}, \qquad\qquad (ii) \quad \ln\, z = (2n+1)\pi\, i \quad n = 0, \pm 1, \pm 2, ...,$$

(b)

$$(i) \quad e^z = -1, \qquad\qquad (ii) \quad e^z = -3.$$

Question 3.9 *Find the image of the annulus*

$$2 < |z| \leq 4,$$

under the principal branch of the logarithmic function. Sketch the graph.

Question 3.10 *Find the image of the sector*

$$1 < Re\ z \leq 2,$$

under the mapping $w = \dfrac{1}{z}$, $z \neq 0$. *Sketch the graph.*

Question 3.11 *Find the image of the line* $Re\ z = 3$, *under the following mappings:*

1. (a)

$$f(z) = \frac{z-3}{z+3},$$

(b)

$$f(z) = e^z.$$

Plot the graphs of the images in `Mathematica`.

Question 3.12 *Find the image of the line Re* $z = 2$, *under the following mappings:*

1. (a)
$$f(z) = z^3,$$
   (b)
$$f(z) = \frac{1}{z+2},$$
   (c)
$$f(z) = ln\frac{z-6}{z+2}.$$

Plot the images with `Mathematica`.

Question 3.13 *Find the fixed points of the mapping*
$$f(z) = \frac{2z-5}{z+4}.$$
Note that: a complex number z *is the fixed point of* $f(z)$ *if* $z = f(z)$.

Question 3.14 *Solve the following equations:*

1. (a)
$$(i) \quad \sin z = 1, \qquad (ii) \quad \cos z = 1.$$
   (b)
$$(i) \quad \sin z = 2, \qquad (ii) \quad \cos z = 2.$$

Question 3.15 *Show that*

1. (a)
$$\sin\ z = \sin\ x\ \cosh\ y + i\cos\ x\ \sinh\ y, \quad z = x + iy.$$
   (b)
$$|\sin\ z| \geq |\sin\ x|, \quad z = x + iy.$$
   (c)
$$(i) \quad |\sin\ z|^2 = \sin^2\ x + \sinh^2\ y, \quad z = x + iy.$$
$$(ii) \quad |\cos\ z|^2 = \cos^2\ x + \sinh^2\ y, \quad z = x + iy.$$
   for $z = x + iy$.

Question 3.16 *Find the region onto which the half complex plane* $Im\ z = y > 0$ *is mapped by the transformation*
$$f(z) = \frac{1+i}{z},$$
by using

1. (a) *Cartesian coordinates*
   (b) *polar coordinates*

Sketch the graph.

Question 3.17 *Find the linear fractional transformation that maps the complex numbers* $z_=2,\ z_2 = i,\ z_3 = -2,$ *onto the numbers* $w_1 = 1,\ w_2 = i,\ w_3 = -1.$

Question 3.18 .

1. (a) *Show that equation*
$$|\frac{z-2}{z+2}| = 4,$$
   represents a circle. Find the center and the radius of this circle. Sketch the graph.
   (b) *Show that the function*
$$w = f(z) = \frac{z-p}{\overline{p}z-1}, \qquad |p| \neq 1,$$
   maps one to one
      i. *the circle* $|z| = 1$ *on the circle* $|w| = 1$,
      ii. *the disc* $|z| < 1$ *on the disc* $|w| < 1$ *if* $|p| < 1$,
      iii. *the disc* $|z| < 1$ *on the set* $|w| > 1$, *if* $|p| > 1$.

Chapter 4

Continuous and Differentiable Functions

4.1 Limits

Let $w = f(z)$ be a function defined in some neighborhood of a number z_0, and not necessary at z_0.

Definition 4.1 *A number g is said to be the limit of $f(z)$ at z_0, if and only if for every $\epsilon > 0$ there exists $\delta_\epsilon(z_0) > 0$, such that, the inequality*

$$0 < |z - z_0| < \delta_\epsilon(z_0)$$

implies the inequality

$$|f(z) - g| < \epsilon.$$

If the limit g exists in the sense of this definition, then we apply the following notation:

$$\lim_{z \to z_0} f(z) = g.$$

We can write the definition in terms of logical quantifies as follows:

$$\forall_{\epsilon>0} \exists_{\delta_\epsilon(z_0)>0} \{0 < |z - z_0| < \delta_\epsilon(z_0) \implies |f(z) - g| < \epsilon\}.$$

Infinite Limit. The limit g of $f(z)$ at z_0 is infinite, if for every $R > 0$ there exists $\delta_R > 0$, such that, the inequality

$$0 < |z - z_0| < \delta_R$$

implies the inequality

$$|f(z)| > R.$$

In symbols, we note

$$\lim_{z \to z_0} f(z) = \infty.$$

Limit in Infinity. A function $f(z)$ has a limit g in infinity, if and only if for every $\epsilon > 0$ there exists $R_\epsilon > 0$, such that, the inequality

$$|z| > R_\epsilon$$

implies the inequality

$$|f(z) - g| < \epsilon.$$

In terms of logical quantifies, we write

$$\forall_{\epsilon>0}\exists_{R_\epsilon>0}\{|z| > R_\epsilon \implies |f(z) - g| < \epsilon\}.$$

In symbols, we note

$$\lim_{z\to\infty} f(z) = g.$$

Example 4.1 *Using definition show that*

$$\lim_{z\to i} \frac{2(z^2+1)}{3(z-i)} = \frac{4}{3}i.$$

Let us note that at $z_0 = i$, the function $f(z) = \dfrac{2(z^2+1)}{3(z-i)}$ is not definite, however $f(z)$ has the limit $g = \dfrac{4}{3}i$. Indeed, we consider $\epsilon > 0$ for which

$$|f(z) - g| = |\frac{2(z^2+1)}{3(z-i)} - \frac{4}{3}i| = \frac{2}{3}|z - i| < \epsilon.$$

Hence, the inequality holds for $|z - i| < \delta_\epsilon = \dfrac{3}{2}\epsilon.$

A limit of a complex valued function $f(z)$ at a point z_0 in `Mathematica` is given by the command:

$$Limit[f(z),\ z- > z_0].$$

For example

```
Limit[(z- I)/(z^2+1),z->I]
```

gives $-\dfrac{I}{2}$, or

```
Limit[2(z^2+1)/(3(z-I)), z->I]
```

gives $\dfrac{4I}{3}$

4.2 Continuity

Let $f(z)$ be a function definite in a neighborhood of a complex number z_0 and also at z_0. Continuity of such a function is considered in the sense of the following definition:

Definition 4.2 *A function $f(z)$ is continuous at z_0 if $f(z)$ has the limit g at z_0 and $g = f(z_0)$.*

Also, we have the $\epsilon - \delta$ definition

Definition 4.3 *A function $f(z)$ is said to be continuous at z_0, if and only if for every $\epsilon > 0$ there exists $\delta_\epsilon(z_0) > 0$, such that, the inequality*

$$0 < |z - z_0| < \delta_\epsilon(z_0)$$

implies the inequality

$$|f(z) - f(z_0)| < \epsilon.$$

In the terms of logical quantifies, we say that a function is continuous at z_0, if and only if the following implication holds:

$$\forall_{\epsilon>0}\exists_{\delta_\epsilon>0}\{0 < |z - z_0| < \delta_\epsilon \implies |f(z) - f(z_0)| < \epsilon\}.$$

Consequently, a function $f(z)$ is continuous in a region, if it is continuous at every complex number of the region.
One can easily show that polynomials, exponential function, sine and cosine are continuous functions on the whole complex plane.
The following theorem holds:

Theorem 4.1 *If $f(z)$ and $g(z)$ are continuous functions then*

$$f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)}, \quad g(z) \neq 0,$$

are also continuous functions.

The proof of this theorem is the same as for real valued functions of a real variable.
Let us note that every function of complex variable can be written in the following form:

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy.$$

Thus, $f(z)$ is a continuous function, if and only the real part $Re\ f(z) = u(x, y)$ and the imaginary part $Im\ f(z) = v(x, y)$ are continuous functions.

4.3 Uniform Continuity

We consider uniform continuity of a function $f(z)$ in a region D in the sense of the following definition:

Definition 4.4 *A function $f(z)$ is uniformly continuous in a region D, if and only if for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$, such that, for any two numbers $z_1 \in D$ and $z_2 \in D$, the inequality*

$$0 < |z_1 - z_2| < \delta_\epsilon$$

implies the inequality

$$|f(z_1) - f(z_2)| < \epsilon.$$

In logical notation, we write

$$\forall_{\epsilon>0}\exists_{\delta>0}\{|z_1 - z_2| < \delta_\epsilon \Longrightarrow |f(z_1) - f(z_2)| < \epsilon\}.$$

Clearly, every function $f(z)$ which is uniformly continuous in a region D is also continuous in the region D, but not vice versa. Let us note that for a uniformly continuous function there exists one $\delta_\epsilon > 0$ independent of location of points z_1 and z_2 in a region.

Example 4.2 *For example, $f(z) = \dfrac{1}{z}$ is continuous in the region*

$$D = \{z \in Z : \ 0 < |z| < 1\}.$$

However, this function is not uniformly continuous in D.

Indeed, for $z_0, z \in D$, we have

$$|f(z) - f(z_0)| = |\frac{1}{z} - \frac{1}{z_0}| = \frac{|z - z_0|}{|zz_0|} < \frac{2|z - z_0|}{|z_0|^2} < \epsilon,$$

for

$$|z - z_0| < \delta_\epsilon(z_0) = \frac{\epsilon}{2}|z_0|^2.$$

Thus, the function is continuous at every point $z_0 \in D$. However, this function is not uniformly continuous in D, since for $z_1 = \dfrac{1}{n+1}$ and $z_2 = \dfrac{1}{n}$, the difference

$$|f(z_1) - f(z_2)| = |(n+1) - n| = 1$$

is not less than small $\epsilon > 0$, in spite of the small distance between the arguments

$$|z_1 - z_2| = |\frac{1}{n+1} - \frac{1}{n}| = \frac{1}{n(n+1)}.$$

for large n.

4.4 Derivatives

Let $f(z)$ be a single valued function defined in a neighborhood of a complex number z_0. Then, the derivative of $f(z)$ at z_0 is defined as the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.1}$$

provided that this limit exists, independently of a path, how z approaches z_0. If the limit exists then $f(z)$ is said to be differentiable at z_0, and its derivative is denoted by $f^{'}(z_0)$ or $\frac{df(z_0)}{dx}$, otherwise, it is referred to as not differentiable function.

Clearly, we can write the limit (4.1) in the equivalent form

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

where $\Delta z = z - z_0$.

Example 4.3 *Let us consider the function*

$$f(z) = \sqrt{1+z}, \quad at \quad z_0 = i.$$

Following the definition, we compute

$$\lim_{z \to i} \frac{\sqrt{1+z} - \sqrt{1+i}}{z - i} = \lim_{z \to i} \frac{z - i}{(z-i)\sqrt{1+z} + \sqrt{1+i}} = \frac{1}{2\sqrt{1+i}} = \frac{1}{\sqrt[4]{32}} e^{-i\frac{\pi}{8}}.$$

Hence, we have

$$\frac{df(z)}{dz}_{|z=i} = \frac{d\sqrt{1+z}}{dz}_{|z=i} = \frac{1}{\sqrt[4]{32}} e^{-i\frac{\pi}{8}}.$$

One can find a derivative of a function $f(z)$ at a point z_0 in `Mathematica` as the limit of the Newton's quotient

$$Limit[\frac{f[z] - f[z_0]}{z - z_0}, \ z- > z_0].$$

For example, let $f(z) = \sqrt{1+z}$. Then, the command

```
Limit[(Sqrt[1+z]-Sqrt[1+I])/(z-I), z->I]
```

gives the derivative $(\frac{1}{4} - \frac{I}{4})\sqrt{1+I} = \frac{1}{\sqrt[4]{32}} e^{-i\frac{\pi}{8}}$.

All rules for derivatives known for real functions are also applicable to complex variable functions.

- **Derivatives of some elementary functions**. Using the definition of a derivative, one can find the following formulae:

$$\frac{dz^n}{dz} = nz^{n-1}, \qquad \frac{de^z}{dz} = e^z,$$
$$\frac{d\sin\ z}{dz} = \cos\ z, \qquad \frac{d\cos\ z}{dz} = -\sin\ z,$$
$$\frac{dtan\ z}{dz} = \frac{1}{\cos^2\ z,} \qquad \frac{dcot\ z}{dz} = -\frac{1}{\sin^2\ z,}$$
$$\frac{d\ln\ z}{dz} = \frac{1}{z}, \qquad \frac{da^z}{dz} = a^z ln\ a.$$

- **Arithmetic operations on derivatives.** Let $f(z)$ and $g(z)$ be differentiable functions in a region D. Then the functions $f(z) \pm g(z)$, $f(z)g(z)$ and $\frac{f(z)}{g(z)}$ are also differentiable in D and their derivative are given by the formulae

$$\frac{d(f(z) \pm g(z)}{dz} = \frac{df(z)}{dz} \pm \frac{dg(z)}{dz},$$
$$\frac{df(z)g(z)}{dz} = g(z)\frac{df(z)}{dz} + f(z)\frac{dg(z)}{dz},$$
$$\frac{d}{dz}\{\frac{f(z)}{g(z)}\} = \frac{f^{'}(z)g(z) - f(z)g^{'}(z)}{g^2(z)}, \qquad g(z) \neq 0.$$

- **The derivative of a composed function.** Let $g(z)$ be a differentiable function at z and $f(w)$ be a differentiable function at $w = g(z)$. Then, the composed function $f(g(z))$ is differentiable at z and its derivative is given by the formula

$$\frac{df(g(z))}{dz} = \frac{df(w)}{dw}\frac{dg(z)}{dz}, \qquad w = g(z).$$

- **The derivative of an inverse function.** Let $w = f(z)$ be a continuous function in a neighbourhood of a point z_0 which maps one to one the neighbourhood of z_0 into neighbourhood of $w_0 = f(z_0)$. If there exits the derivative $f^{'}(z_0) \neq 0$ then the inverse function $z = f^{-1}(w)$ has derivative at w_0 given by the formula

$$\{f^{(-1)}(w_0)\}^{'} = \frac{1}{f^{'}(z_0)}, \qquad w_0 = f(z_0).$$

In general, derivatives in `Mathematica` are given by the following commands:

```
D[f[z],z];   D[f[z],{z,n}];   Dt[f[z[t]],t];
```

where D[f[z],z] stands for the first derivative, D[f[z],{z,n}] stands for the derivative of order n, and D[f[z[t]],t] stands for the total derivative.
For example, let $f(z) = \sqrt{1+z}$. Then, the commands

```
f[z_]:=Sqrt[1+z];
         D[f[z],z]
```

give the derivative $\frac{1}{2\sqrt{1+z}}$, and the commands

```
f[z_]:=Sqrt[1+z];
        D[f[z],{z,2}]
```

give the derivative $-\frac{1}{4\sqrt{1+z}^{3/2}}$, and the commands

```
f[z_]:=Sqrt[1+z];
z[t_]:=2(Cos[t]+I Sin[t]);
        D[f[z],z]
```

give the total derivative

$$\frac{I\cos[t] + \sin[t]}{\sqrt{1 + 2(\cos[t] + I\sin[t])}}$$

,

4.5 Exercises

Question 4.1 *Use the definition to show that*

1. *(a) the function*

$$f(z) = \frac{z^2+4}{z^3 - 2z^2 + 4z - 8}$$

has the limit $g = \frac{-1+i}{4}$, *at* $z_0 = -2i$.

(b) the function

$$f(z) = \frac{\overline{z}}{z}$$

does have a limit at $z_0 = 0$.

Question 4.2 *Let* $f(z) = 3z^2 + 2z$. *Use the definition to show that*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 6z_0 + 2,$$

at any point z_0

Question 4.3 *Find the limit*

1. (a)
$$\lim_{z\to 1-i}[x+i(2x+y)].$$
(b)
$$\lim_{z\to \frac{\pi}{2}} \frac{\sin z}{z}, \qquad Anw:\ \frac{2}{\pi}$$
(c)
$$\lim_{z\to \frac{i\pi}{2}} z^2 \cosh\frac{4z}{3}, \qquad Anw:\ \frac{\pi^2}{8}$$

Question 4.4 *Use the definition to show that the function*
$$f(z) = z^4 + z^2 + 1$$
is uniformly continuous in the disc $|z| \le R$.

Question 4.5 *Show that the function*
$$f(z) = \frac{1}{z^2}$$
is not uniformly continuous in the disc $|z| \le R$*, but it is uniformly continuous in the annulus* $\frac{R}{2} \le |z| \le R$.

Question 4.6 *Use the definition of a derivative to show that the functions* $f(z) = \overline{z} - 2,\ Re\ z$ *and* $g(z) = Im\ z$ *are nowhere differentiable.*

Question 4.7 *Show that the function* $f(z) = |z|^2$ *is differentiable at* $z_0 = 0$*, but it is not differentiable at any point* $z_0 \neq 0$.

Question 4.8 *Let*
$$f(z) = \begin{cases} \dfrac{\sin\ z}{z}, & z \neq 0, \\ 0, & z = 0 \end{cases}$$
Show that the function $f(z)$ *is differentiable on whole complex plane.*

Question 4.9 *Let* $f(z) = u(x,y) + iv)x,y)$ *has derivative* $f'(z)$*. Show that the function* $g(z) = u(x,y) - iv(x,y)$ *has derivative* $g'(z)$ *if and only* $f'(z) = 0$.

Chapter 5

Analytic Functions

5.1 Cauchy Riemann Equations

Let us consider a complex variable function in the following form:

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy.$$

Suppose that $f(z)$ is differentiable function, that is, there exists limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f^{'}(z), \qquad \Delta\, z = z - z_0,$$

and it is independent on a path through which z approaches z_0, $(\Delta\, z \to 0)$
Choosing the path along the x axis, we compute

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = & \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\
& + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
& = \frac{\partial u(x_0, y_0)}{\partial x} + i \frac{\partial v(x_0, y_0)}{\partial x}.
\end{aligned}$$

Similarly, choosing the path along y axis, we compute

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = & \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i \Delta y} \\
& + i \lim_{\Delta y \to 0} \frac{v(x_0, x, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y} \\
& = -i \frac{\partial u(x_0, y_0)}{\partial y} + \frac{\partial v(x_0, y_0)}{\partial y}.
\end{aligned}$$

Comparing the right hand sides

$$\frac{\partial u(x_0, y_0)}{\partial x} + i \frac{\partial v(x_0, y_0)}{\partial x} = -i \frac{\partial u(x_0, y_0)}{\partial y} + \frac{\partial v(x_0, y_0)}{\partial y}.$$

we arrive at the Cauchy-Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned} \tag{5.1}$$

Example 5.1 *The exponential function e^z, $z = x + iy$, satisfies Cauchy-Riemann equations, since*

$$e^z = e^x(\cos\ y + i\sin\ y) = u(x,y) + iv(x,y),$$

where

$$u(x,y) = e^x \cos\ y, \qquad v(x,y) = e^x \sin\ y.$$

Clearly, we have

$$\frac{\partial u}{\partial x} = e^x \cos\ y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin\ y = -\frac{\partial v}{\partial x}$$

In this way, we have proved the following theorem

Theorem 5.1 *If a function*

$$f(z) = u(x,y) + iv(x,y), \qquad z = x + iy,$$

possesses derivative $f^{'}(z)$, then functions $u(x,y)$ and $v(x,y)$ satisfy Cauchy-Riemann equations.

However, there are complex variable functions which satisfy Cauchy-Riemann equations and are not differentiable.

Example 5.2 *The function*

$$f(z) = \sqrt{|xy|} + i\ xy = u(x,y) + iv(x,y),$$

satisfies Cauchy-Riemann equations at $z = 0$, since

$$\frac{\partial u(x,0)}{\partial x} = \frac{\partial v(0,y)}{\partial y} = 0$$

$$\frac{\partial u(0,y)}{\partial y} = -\frac{\partial v(x,0)}{\partial x} = 0$$

However, the derivative of $f(z)$ at $z = 0$ does not exists.

Indeed, for $x = \alpha t$ and $y = \beta t$, we have Newton's quotient

$$\frac{f(z) - f(0)}{z} = \frac{\sqrt{|xy|}}{x + iy} + i\frac{xy}{x + iy} = \pm\frac{\sqrt{\alpha\beta}}{\alpha + i\beta} + i\,t\frac{\alpha\beta}{\alpha + i\beta}.$$

Thus, the limit

$$\lim_{t\to 0}(\pm\frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}+i\,t\frac{\alpha\beta}{\alpha+i\beta})=\pm\frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}.$$

This limit depends on a path along which z approaches zero. Because for different α or β the limit attains different values, therefore, the derivative $f^{'}(0)$ does not exist at $z=0$.
The following theorem holds:

Theorem 5.2 *If the functions $u(x,y)$ and $v(x,y)$ have continuous partial derivatives*

$$\frac{\partial u}{\partial x},\quad \frac{\partial u}{\partial y},\quad \frac{\partial v}{\partial x},\quad \frac{\partial v}{\partial y},$$

in a neighborhood of a complex number z and if Cauchy-Riemann equations hold at z, then the function $f(z)$ possesses the derivative $f^{'}(z)$ at z.

Thus, Cauchy-Riemann equations are equivalent to differentiability of $f(z)=u(x,y)+iv(x,y)$, provided that the partial derivative $u_x(x,y), v_x(x,y), u_y(x,y)$ and $v_y(x,y)$ are continuous functions.
The following module checks whether or not a function $f(z)=u(x,y)+iv(x,y)$ satisfies Cauchy Riemann's equations:

```
  cauchyRiemann[u_,v_]:=Module[{ux,uy,vx,vy },
ux=D[u,x];
vx=D[v,x];
uy=D[u,y];
vy=D[v,y];
(ux===vy)And(uy===-vx)
       ]
```

For example, for $f(z)=x^2-y^2+2i\,xy$, input $u(x,y)$ and $v(x,y)$ and activate the module by the commands

```
u=x^2-y^2;
v=2 x y;
cauchyRiemann[u,v]
```

to obtain the output `True`.

5.2 Definition of Analytic Functions

The class of analytic functions is determined by the following definition:

Definition 5.1 .

- *A function $f(z)$ is said to be analytic at a complex number z if $f(z)$ possesses the derivative $f'(z)$ at z and in a neighborhood of z.*

- *A function $f(z)$ is analytic in a domain D if it is analytic at each complex number of D.*
- *A function $f(z)$ is analytic in $z = \infty$ if $f(\frac{1}{z})$ is analytic at $z = 0$.*

Later on, we shall show that if an analytic function $f(z)$ possesses its first derivative then $f(z)$ possesses all derivatives, that is, any analytic function is infinite times differentiable.
In the class of analytic functions, there are two important of subclasses

- Entire functions.
- Harmonic functions.

These subclasses are defined as follows:

Definition 5.2 $f(z)$ *is called entire function if it is analytic in whole complex plane.*

Definition 5.3 $f(z) = u(x, y) + iv(x, y)$ *is called harmonic function if its real part $u(x, y)$ and imaginary part $v(x, y)$ are harmonic functions, that is, if they satisfy Laplace's equations*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Now, let us show that real part $u(x, y)$ and imaginary part $v(x, y)$ of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic functions. Indeed, these functions satisfy Cauchy-Riemann equations

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y},$$

$$\frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x}.$$

By differentiation of the first equation with respect to x, and the second equation with respect to y, we get

$$\frac{\partial^2 u(x, y)}{\partial x^2} = \frac{\partial^2 v(x, y)}{\partial x \partial y}$$

$$\frac{\partial^2 u(x, y)}{\partial y^2} = -\frac{\partial^2 v(x, y)}{\partial y \partial x}$$

Hence, we have

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0.$$

Let us note that if $f(z)$ is an analytic function then the derivative

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and also

$$\frac{df}{dz} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Thus, any analytic function satisfies the following partial differential equation:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}, \tag{5.2}$$

which implies the real Cauchy-Riemann equations. Let $f(x,y)$ be a complex valued function of two real variables x and y. Clearly, for a complex number $z = x + iy$, we have $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$. So that, we can consider $f(x(z,\overline{z}), y(z,\overline{z}))$ as a function of two variables z and $\overline{z}$. Applying the rule of differentiation of a composed function, we compute

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Hence, by equation (5.2), we obtain the following necessary condition to be the function f analytic

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This means an analytic function is independent of $\overline{z}$.
The following module checks the necessary condition of anslysity of a function $f(z) = u(x,y) + iv(x,y)$,

```
analyticCondition[u_,v_]:=Module[{z,s,f ,pu,pv},
        pu=u[(z+s)/2,(z-s)/(2*I)];
        pv=v[(z+s)/2,(z-s)/(2*I)];
        f=pu+I*pv;
        Simplify[D[f,s]]===0
      ]
```

For example, let $f(z) = z^2$. Then, input data functions

```
u[x_,y_]:=x^2-y^2; v[x_,y_]:=2*x*y;
```

and execute the module

```
analyticCondition[u,v]
```

to obtain the answer `True`.

5.3 Liouville's Theorem

Let us state Liouville's theorem for entire functions.

Theorem 5.3 *If $f(z)$ is an entire function bounded on the complex plane, that is, there exists a generic constant M such that*

$$|f(z)| \leq M, \qquad z \in Z,$$

then $f(z)$ is a constant function throughout the whole complex plane.

Proof. $f(z)$ as an entire function satisfies Cauchy's inequality

$$|f^{(n)}(z)| \leq \frac{n!M}{R^n}, \qquad n = 0, 1, ...,$$

for $|z| \leq R$ and any radius R, where M is a constant independent of R.
For $n = 1$, we have

$$|f'(z)| \leq \frac{M}{R}.$$

Hence, when $R \to \infty$, we get $f'(z) \equiv 0$ for $z \in Z$. This means that $f(z) =$ *constant* on the whole complex plane.
As an implication of Liouville's theorem, we shall prove the Fundamental Theorem of Algebra.

5.4 Fundamental Theorem of Algebra

Theorem 5.4 *Every polynomial*

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

of degree $n \geq 1$ has at least one zero on complex plane.

Proof. Proving by contradiction, suppose that $P_n(z) \neq 0$ for all z on the complex plane Z. This means that the function

$$f(z) = \frac{1}{P_n(z)},$$

is analytic on the whole complex plane, that is, $f(z)$ is an entire function. Now, let us show that $f(z)$ is a bounded function on the whole complex plane. Namely, we have

$$\frac{1}{|z|^n}|a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z^|} \leq \frac{|a_n|}{2}$$

and

$$|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}| \leq \frac{|a_n|\ |z|^n}{2}$$

for all $|z| \geq R$ and sufficiently large R.
To estimate $f(z)$, we write

$$|f(z)| = \frac{1}{|P_n(z)|} \leq \frac{1}{|a_n|\ |z|^n - |a_0 + a_1 z + \cdots + a_{n-1}|z^{n-1}|} \leq \frac{2}{|a_n|\ |z|^n} \leq \frac{2}{|a_n|\ R^n}$$

for all $|z| \geq R$.
Thus, $f(z)$ is a bounded by $\dfrac{2}{|a_n|\ |R^n}$ in the region $|z| \geq R$ and as an analytic function is also bounded by a constant M_R in the disc $|z| \leq R$. Thus, $f(z)$ is bounded by the total constant

$$M = \max\{M_R, \frac{2}{|a_n|\ |R|^n}\},$$

so that

$$|f(z)| \leq M, \qquad \text{for all} \quad z \in Z.$$

By Liouville's theorem, $f(z)$ is a constant function. This contradicts the assumption $P_n(z) \neq 0$ for all $z \in Z$. Thus, $\dfrac{1}{P_n(z)}$ is not an entire function and $P_n(z)$ has at least one zero on complex plane. The end of the proof.

5.5 Maximum Modulus Principle

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function inside and on a closed curve C then the maximum value of $|f(z)|$ is attainable at a point $z_0 \in C$, that is

$$\max_{z \in D \cup \in C} |f(z)| = |f(z_0)|,$$

for a certaint point $z_0 \in C$.
Proof. Let

$$M(x,y) = |f(z)|^2 = u^2(x,y) + v^2(x,y), \qquad z = x + iy,$$

be the square of the modulus of $f(z)$. We shall show that function $M(x,y)$ satisfies the differential inequality

$$\frac{\partial^2 M(x,y)}{\partial x^2} + \frac{\partial^2 M(x,y)}{\partial y^2} \geq 0, \tag{5.3}$$

in the region D enclosed by the curve C. Indeed, we have

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = 2[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2] \geq 0.$$

From the inequality (5.3) it follows that $M(x,y)$ does not attains its maximum inside of C unless it is an constant function.
End of the proof.

5.6 Exercises

Question 5.1 .

1. (a) *Find Cauchy-Riemann equations of an analytic function* $f(z) = u(x, y) + iv(x, y)$ *in the polar coordinates*

$$x = r\cos\,\theta, \qquad y = r\sin\,\theta, \qquad z = x + iy \neq 0.$$

   (b) *Check whether or not the function* $f(z) = z\,e^z$ *satisfies Cauchy-Riemann equations in polar coordinates.*

Question 5.2 *Show that a harmonic function* u *satisfies the following formal differential equation*

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

Question 5.3 *Determine the coefficients* a, b, c *and* d, *to be*

1. (a) *the quadratic polynomial*

$$ax^2 + bxy + cy^2$$

   an entier function.

   (b) *the cubic polynomial*

$$ax^3 + bx^2y + cxy^2 + dy^3$$

   an entier function.

Question 5.4 *Check weather or not the following functions are entire and satisfy the Cauchy-Riemann equations:*

1. (a)

$$f(z) = xy^2 + ix^2$$

   (b)

$$f(z) = e^y e^{ix}.$$

Question 5.5 *Let* $f(z)$ *be an anslytic function in a region* D. *Show that if* $f(z)$ *is real valued function in* D *then* $f(z) = constant$ *in the region* D.

Question 5.6 *Show that the following functions are entire:*

1. (a)

$$f(z) = e^{-y}e^{ix}, \qquad z = x + iy,$$

   (b)

$$f(z) = (z^2 - 2)e^{-x}e^{-iy}, \qquad z = x + iy.$$

Question 5.7 *Let* $f(z) = u(r,\theta) + iv(r,\theta), \quad z = r(\cos\ \theta + i\sin\ \theta) \in D$ *be analytic function in a domain* D *which does not include number* $z = 0$*. Using Cauchy-Riemann equations in polar coordinates, show that function* $u(r,\theta)$ *satisfies Laplace's equation in polar coordinates equation*

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Also, show that function $v(r,\theta)$ *satisfies Laplace's equation in polar coordinates.*

Question 5.8 *Consider the following function*

$$f(z) = z\ e^z, \qquad z = x + iy.$$

Show that the real part $u(x,y) = Re\ f(z)$ *and the imaginary part* $v(x,y) = Im\ f(z)$ *satisfy the following equations:*

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = 0, \qquad \frac{\partial^2 v}{\partial z \partial \overline{z}} = 0.$$

Question 5.9 *Consider the following function*

$$f(z) = x^2 - y^2 + 2ixy, \qquad z = x + iy.$$

Show that this function is analytic and satisfies the equation

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Question 5.10 *Following details of the proof of fundamental theorem of algebra, show that the particular polynomial*

$$P_4(z) = z^4 - z^2 - 2z + 2$$

has exactly four roots. Determine the roots.

Question 5.11 *Find the maximum of* $|f(z)|$ *in the disc* $|z| \leq 1$*, when* $f(z) = z^4 + z^2 + 1$.

Question 5.12 *Show that for every polynomial*

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \qquad a_n \neq 0, \quad n \geq 1,$$

there exists a number $R > 0$ *such that*

$$|P_n(z)| > \frac{|a_n|\ |z|^n}{2},$$

for every $|z| > R$.

Question 5.13 *Show that every harmonic function in a domain* D *is either constant or does not attain its positive maximum or negative minimum in* D.

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Chapter 6

Integrals

6.1 The Integral of a Complex Valued Function of a Real Variable

Let us consider the following complex valued function:

$$w(t) = u(t) + iv(t),$$

in real the variable $t \in [\alpha, \beta]$.
Assuming that the real functions $u(t)$ and $v(t)$ are integrable in the interval $[\alpha, \beta]$, we define the integral of the complex valued function $w(t)$ by the formula

$$\int_{\alpha}^{\beta} w(t)dt = \int_{\alpha}^{\beta} u(t)dt + i\int_{\alpha}^{\beta} v(t)dt.$$

Example 6.1 *Let* $w(t) = \cos\ t + i \sin\ t,\ \ 0 \leq t \leq \dfrac{\pi}{2}.$

We have

$$\int_{0}^{\frac{\pi}{2}} w(t)dt = \int_{0}^{\frac{\pi}{2}} \cos\ t\ dt + i\int_{0}^{\frac{\pi}{2}} \sin\ t\ dt = 1 + i.$$

The following inequality holds:

$$|\int_{\alpha}^{\beta} w(t)dt| \leq \int_{\alpha}^{\beta} |w(t)|dt \tag{6.1}$$

Indeed, if

$$\int_{\alpha}^{\beta} w(t)dt = 0$$

then inequality is true.
If

$$\int_{\alpha}^{\beta} w(t)dt \neq 0$$

then there exist real $r_0 \neq 0$ and θ_0 such that

$$\int_{\alpha}^{\beta} w(t)dt = r_0 e^{i\theta_0}.$$

Hence, we have

$$r_0 = \int_{\alpha}^{\beta} e^{-i\theta_0} w(t)dt$$

and

$$r_0 = \int_{\alpha}^{\beta} Re[e^{-i\theta_0} w(t)]dt$$

But

$$Re[e^{-i\theta_0} w(t)] \leq |e^{-i\theta_0} w(t)| = |w(t)|.$$

Thus

$$|\int_{0}^{\beta} w(t)dt| \leq \int_{\alpha}^{\beta} |w(t)|dt.$$

One can integrate a complex valued function $g(t)$ of real variable t by the following Mathematica command:

```
Integrate[g[t],{t,a,b}]
```

For example, executing the commands

```
g[t_]:=t^2+I*Sin[Pi t];
Integrate[g[t],{t,0,1}]
```

we obtain the value of the integral

$$\frac{1}{3} + \frac{2I}{\pi}.$$

6.2 Line Integrals

Let $f(z)$ be a function of complex variable $z = x + iy$ defined on a contour C given by the parametric equation

$$C: \quad z(t) = x(t) + iy(t), \quad \alpha \leq t \leq \beta.$$

The line integral of $f(z)$ along the contour C is defined as

$$\int_C f(z)dz = \int_{\alpha}^{\beta} f(z(t))z^{'}(t)dt. \tag{6.2}$$

If $f(z) = u(x, y) + iv(x, y), \quad z'(t) = x'(t) + iy'(t)$ then the line integral takes the following form:

$$\int_C f(z)dz = \int_{\alpha}^{\beta} (ux' - vy')dt + i \int_{\alpha}^{\beta} (vx' + uy')dt.$$

In terms of differentials $dx = x'dt$ and $dy = y'dt$, this integral is

$$\int_C f(z)dz = \int_{\alpha}^{\beta} (udx - vdy) + i \int_{\alpha}^{\beta} (vdx + udy).$$

Example 6.2 *Compute the line integral*

$$\int_C \frac{dz}{z - z_0},$$

along the circle $C: \quad z(t) = z_0 + Re^{it}, \quad 0 \le t < 2\pi.$

We have

$$f(z) = \frac{1}{z - z_0}, \qquad z'(t) = iRe^{it},$$

By formula (6.2), we compute

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{Rie^{it}}{Re^{it}} dt$$

$$= \int_0^{2\pi} idt = 2\pi\, i.$$

Example 6.3 *Let us consider the line integral*

$$N(C, a) = \frac{1}{2\pi i} \int_C \frac{dz}{z - a},$$

along a closed contour $C\ : z = z(t) \quad \alpha \le t \le \beta, .$ *Then,* $N(C, a)$ *is an integer number.*

Let us note that

$$N(C, a) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{z^{'}(t)\, dt}{z(t) - a}.$$

We shall show that $N(C, a)$ is an integer number which depends on allocation of a with respect to the curve C.
Indeed, the function

$$g(t) = \int_\alpha^t \frac{z^{'}(s)}{z(s) - a} ds,$$

has the derivative

$$g^{'}(t) = \frac{z^{'}(t)}{z(t) - a}.$$

Thus, the function

$$G(t) = e^{-g(t)}[z(t) - a], \qquad \alpha \le t \le \beta.$$

has the derivative

$$\frac{dG(t)}{dt} = \frac{d}{dt}[e^{-g(t)}(z(t) - a)] = 0, \qquad \alpha \le t \le \beta.$$

and $G(t) = constant$ in the interval $[\alpha, \beta]$. Because $g(\alpha) = 0$, therefore

$$e^{-g(t)}[z(t) - a] = z(\alpha) - a.$$

Hence, we find

$$e^{g(t)} = \frac{z(t) - a}{z(\alpha) - a}.$$

and

$$e^{g(\beta)} = 1, \qquad \alpha \le t \le \beta,$$

when $z(\beta) = z(\alpha)$.
By the theorem 3.1, we obtain $g(\beta) = 2\pi\ i\ N$. Thus, we find

$$N(C, a) = \frac{g(\beta)}{2\pi\ i} = N.$$

for an integer N.
The number $N(C, a)$ is called index of the point a with respect to the curve C. This index indicates how many times point $z(t)$ passes around a when t moves from α to β. For example. if C is the circle $z(t) = a + Re^{it}, \quad 0 \le t \le 2\pi$, then $N(C, a) = 1$, since then point $z(t)$ moves once around the center a when t changes from 0 to 2π.
In order to evaluate a contour integral along a piecewise linear path with vertices $z_1, z_2, ..., z_m$, in `Mathenatica`, we execute the command

$$Integrate[g[z], \{z, z_1, z_2, ..., z_m\}]$$

For example, executing the commands

```
g[z_]:=z*Exp[z^2];
Integrate[g[z],{z,0,I,1+I}]
```

we obtain the value of the integral

$$-\frac{1}{2} + \frac{1}{2}E^{2I}.$$

Properties of Line Integrals.

1. (a) From definition of the line integral the following additive properties can be easily established:

   i.
   $$\int_C (f(z) \pm g(z))dz = \int_C f(z)dz \pm \int_C g(z)dz.$$

   ii.
   $$\int_C (f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

   where the contour C consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3.

(b)
$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

where the contour C has positive orientation, that is, if t moves from α to β then $z(t) = x(t) + iy(t)$ moves from $z_1 = z(\alpha) + iy(\alpha)$ to $z_2 = z(\beta) + iy(\beta)$, and the contour $-C$ has opposite orientation to C.
Indeed, the parametric equation of the contour $-C$ is

$$z(-t) = x(-t) + iy(-t), \qquad -\beta \leq t \leq -\alpha.$$

Then, by the substitution $s = -t$, we have

$$\int_{-\beta}^{-\alpha} f(z(-t))z'(-t)dt = -\int_{\alpha}^{\beta} f(z(s))z'(s)ds = -\int_C f(z)dz.$$

(c)
$$|\int_C f(z)dz| \leq \int_C |f(z)|dz$$

By definition (6.2) and inequality (6.1), we have

$$|\int_C f(z)dz| = |\int_{\alpha}^{\beta} f(z(t))z'(t)dt| \leq \int_{\alpha}^{\beta} |f(z(t))z'(t)|dt = \int_C |f(z)|dz$$

(d) Let $|f(z)| \leq M$ be bounded function on contour C. Then

$$|\int_C f(z)dz| \leq ML,$$

where L is the length of C.
Since, we have

$$|\int_C f(z)dz| \leq |\int_{\alpha}^{\beta} f(z(t))z'(t)|dt| \leq M\int_{\alpha}^{\beta} |z'(t)|dt \leq ML.$$

6.3 Antiderivative

Let $f(z)$ be a function given in the domain D. Every differentiable function $F(z)$ in domain D which satisfies the condition $F'(z) = f(z),\ z \in D$, is called antiderivative of function $f(z)$. Let us note that the antiderivative $F(z)$ is not uniquely determined since if $F(z)$ is an antiderivative then $F(z) + constant$ is also an antiderivative of $f(z)$.
The following theorem holds:

Theorem 6.1 *Let $f(z)$ be a continuous function throughout a domain D and let $F(z)$ be antiderivative to $f(z)$, that is,*

$$F'(z) = f(z), \qquad z \in D.$$

Suppose that a contour $C: \; z(t) = x(t) + iy(t), \;\; \alpha \le t \le \beta$ with the end points $a = z(\alpha)$ and $b = z(\beta)$ lies within the domain D. Then, the line integral is independent of the path C and its value is given by the formula

$$\int_C f(z)dz = \int_a^b f(z)dz = F(b) - F(a).$$

Proof. Let us the define function $\Phi(t) = F(z(t)), \; \alpha \le t \le \beta$. This function is differentiable and the derivative

$$\Phi'(t) = f(z(t))z'(t), \qquad \alpha \le t \le \beta.$$

Applying the formula (6.2), we obtain the formula

$$\int_C f(z)dz = \int_\alpha^\beta f(z(t))z'(t)dt = \int_\alpha^\beta \Phi'(t)dt = \Phi(\beta) - \Phi(\alpha) = F(b) - F(a).$$

Example 6.4 *The function $f(z) = (z - z_0)^n$ has the antiderivative*

$$F(z) = \frac{(z - z_0)^{n+1}}{n+1}, \qquad n \ne -1.$$

By the thesis of the theorem

$$\int_C (z - z_0)^n dz = \frac{1}{n+1}[(b - z_0)^{n+1} - (a - z_0)^{n+1}], \qquad n \ne -1,$$

In particular, if the contour C is close, that is, when $a = b$, the integral

$$\int_C (z - z_0)^n dz = 0, \qquad n \ne -1,$$

6.4 Cauchy Theorem

Now, let us state Cauchy theorem.

Theorem 6.2 *If $f(z)$ is an analytic function within and on a closed contour C, then*

$$\int_C f(z)dz = 0.$$

Example 6.5 *If C is the circle $|z| = 1$ then the integral*

$$\int_C \frac{dz}{z^2 + 2z + 2} = 0.$$

Since then the singular points $z_1 = -1 - i$ and $z_2 = -1 + i$ of the integrand

$$f(z) = \frac{1}{z^2 + 2z + 2} = \frac{i}{2}[\frac{1}{z+1+i} - \frac{1}{z+1-i}]$$

are outside of the circle. Therefore, by Cauchy theorem, the integral along C is equal to zero.
If C_0 is the circle $|z+1-i| = 1$, then the $f(z)$ is not an analytic function in the disc $D = \{z \in Z : \ |z+1-i| \leq 1\}$. Then, Cauchy theorem is not applicable in D. However, we can compute the integral (see (6.2))

$$\int_{C_0} \frac{dz}{z^2+2z+2} = \int_{C_0} \frac{i}{2}[\frac{1}{z+1+i} - \frac{1}{z+1-i}]dz = \pi.$$

Let us note, that Cauchy theorem can be confirmed in `Mathematica` to show that the integral of an analytic function $f(z)$ along a closed contour C is equal to zero. For example, executing the commands

```
g[z_]:=z*Exp[z^2];
Integrate[g[z],{z,0,I,1+I,0}]
```

we obtain the value of the integral equal to zero.

6.5 Cauchy Integral Formula

Let $f(z)$ be an analytic function within and on a closed contour C which is positively oriented, that is, oriented in counterclockwise direction.
Then the following Cauchy Integral Formula holds:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta, \tag{6.3}$$

for any z interior to C.
Proof. Let z be a complex number interior to C and let C_r be the circle with the center at z and radius r which is inside of C and has positive orientation. Then, the function

$$\frac{f(\zeta)}{\zeta - z}, \qquad \zeta \in C \cup C_r,$$

is analytic in the domain D bounded by contour C and circle C_r. Therefore, by Cauchy theorem

$$\int_{C\cup C_r} \frac{f(\zeta)}{\zeta - z}d\zeta = 0.$$

Hence, we have

$$\int_C \frac{f(\zeta)}{\zeta - z}d\zeta - \int_{C_r} \frac{f(\zeta)}{\zeta - z}d\zeta = 0.$$

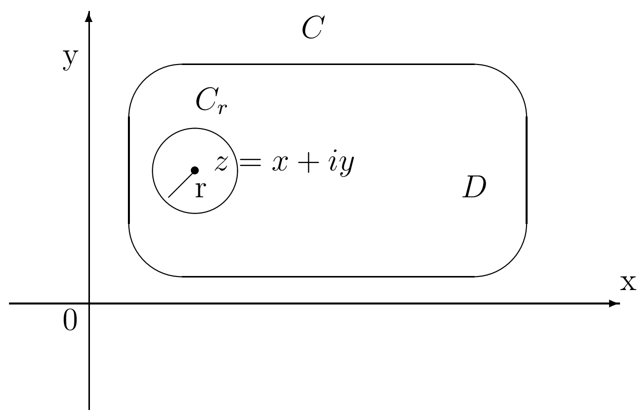

Fig. 6.1 Domain D

Thus, the integral along curve C is equal to the integral along the circle C_r, that is,

$$\int_C \frac{f(\zeta}{\zeta - z} d\zeta = \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Using the equality

$$\int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_r} \frac{f(z)}{\zeta - z} d\zeta + \int_{C_r} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta. \tag{6.4}$$

we obtain

$$\int_{C_r} \frac{f(z)}{\zeta - z} d\zeta = 2\pi i f(z), \qquad z \in D. \tag{6.5}$$

Because $f(z)$ is an analytic function, therefore for every $\epsilon > 0$ there exists $r > 0$ such that

$$|f(\zeta) - f(z)| < \frac{\epsilon}{2\pi}, \quad \text{if} \quad |\zeta - z| < r.$$

Hence, we get the following $\epsilon-$estimate of the integral

$$|\int_{C_r} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta| \leq \frac{\epsilon}{2\pi} |\int_{C_r} \frac{d\zeta}{\zeta - z}| < \epsilon, \tag{6.6}$$

provided that $|\zeta - z| < r$.
Combining equalities (6.4) and the inequality (6.6), we obtain the Cauchy Integral Formula.

Example 6.6 *Let us evaluate the integral*

$$\int_C \frac{z}{1 + z^2} \, dz, \quad C \, : \, |z - \frac{i}{2}| = 1$$

using Cauchy Integral Formula.

The integrand
$$\frac{z}{1+z^2} = \frac{z}{(z-i)(z+i)},$$
has the singular point $z = i$ within the circle $C \; : \; |z - \frac{i}{2}| = 1$. Thus, the function
$$f(z) = \frac{z}{z+i},$$
is analytic within and on the circle C. By Cauchy Integral Formula
$$\int_C \frac{\zeta \, d\zeta}{(\zeta+i)(\zeta-i)} = \int_C \frac{f(\zeta)}{\zeta - i} d\zeta = 2\pi i f(i) = \pi i.$$
Hence, we obtain
$$\int_C \frac{z \, dz}{1+z^2} = \pi \; i.$$

6.6 Cauchy Integral Formula

Theorem 6.3 *Let $f(z)$ be an analytic function within and on a closed contour C positively oriented, then the following Cauchy formula holds:*
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta, \qquad n = 0, 1, ..., \tag{6.7}$$
for any complex number z interior to C.

Proof. We shall prove the theorem using principle of mathematical induction. The theorem is true for $n = 0$, since it is the case of Cauchy Integral Formula which has been already proved.
Assuming that the formula is true for $n = k$, we shall show that the formula is also true for $n = k+1$. Indeed, by the assumption, we have
$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta.$$
Now, let us consider the Newton quotient
$$\begin{aligned} \frac{f^{(k)}(z+\Delta z) - f^{(k)}(z)}{\Delta z} &= \frac{k!}{2\pi i \Delta z} \int_C f(\zeta)[\frac{1}{(\zeta - z - \Delta z)^{k+1}} - \frac{1}{(\zeta - z)^{k+1}}] d\zeta \\ &= \frac{k!}{2\pi i} \int_C f(\zeta) \frac{(\zeta - z)^{k+1} - (\zeta - z - \Delta z)^{k+1}}{\Delta z (\zeta - z - \Delta z)^{k+1} (\zeta - z)^{k+1}} d\zeta. \end{aligned}$$
Using the limit
$$\lim_{\Delta z \to 0} \frac{(\zeta - z)^{k+1} - (\zeta - z - \Delta z)^{k+1}}{\Delta z} = (k+1)(\zeta - z)^k,$$

one can show that

$$f^{(k+1)}(z) = \lim_{\Delta z\to 0} \frac{f^{(k)}(z+\Delta z) - f^{(k)}(z)}{\Delta z} = \frac{(k+1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{k+2}}\, d\zeta.$$

As a consequence of this theorem, the following corollary holds:

Corolary 6.1 *If a function $f(z)$ possesses first derivative $f'(z)$ in a domain D, then $f(z)$ possesses all derivatives in D.*

Example 6.7 *Use Cauchy Integral Formulas to evaluate the integrals*

$$(i) \quad \int_{|z-i|=2} \frac{dz}{z^2+4}, \qquad (ii) \quad \int_{|z-i|=2} \frac{dz}{(z^2+4)^2}.$$

Let us note that we can write the integrals as follows

$$(i) \quad \int_{|z-i|=2} \frac{dz}{(z+2i)(z-2i)}, \qquad (ii) \quad \int_{|z-i|=2} \frac{dz}{(z+2i)^2)(z-2i)^2}.$$

Choosing $f(z) = \dfrac{1}{z+2i}$, we can write the first integral as

$$f(z) = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Thus, for $z = 2i$, we have

$$f(2i) = \frac{1}{4i} = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{d\zeta}{(\zeta+2i)(\zeta-2i)} = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{dz}{z^2+4}.$$

Hence, the first integral is

$$\int_{|z-i|=2} \frac{dz}{z^2+4} = \frac{\pi}{2}.$$

Similarly, choosing $f(z) = \dfrac{1}{(z+2i)^2}$, we have

$$f(z) = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Thus

$$f(z) = \frac{1}{(z+2i)^2} = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{d\zeta}{(\zeta+2i)^2(\zeta - z)}.$$

and the derivative

$$f^{'}(z) = \frac{-2}{(z+2i)^3} = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{d\zeta}{(\zeta+2i)^2(\zeta - z)^2}.$$

Hence, for $z = 2i$, we obtain

$$\frac{-i}{32} = \frac{1}{2\pi i}\int_{|z-i|=2} \frac{d\zeta}{(\zeta+2i)^2(\zeta-z)^2} = \int_{|z-i|=2} \frac{d\zeta}{(\zeta^2+4)^2},$$

and the second integral

$$\int_{|z-i|=2} \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}.$$

Also, one can evaluate, in `Mathematica`, a contour integral of a function $f(z)$ which has singular points interior to a closed contour C. For example, executing the following commands:

```
g[z_]:=z/(z^2+4);
Simplify[Integrate[g[z],{z,-1,1,4 I,-1}]]
```

we obtain the value $2\pi\, i$ of the integral

$$\int_C \frac{z}{z^2+4} dz,$$

along the polygon C with vertices $-1,\ 1,\ 4i, -1$, and with the singular point $z_1 = 2i$ interior to C.

6.7 Cauchy Inequality

If $f(z)$ is analytic inside and on the circle $C:\ |z-a| = R$ then the following Cauchy inequality holds:

$$|f^{(n)}(a)| \le \frac{M_R\, n!}{R^n}, \quad n = 0, 1, ..., \tag{6.8}$$

where $M_R = \max_{|z-a|=R} |f(z)|$.
Indeed, by the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta, \qquad n = 0, 1, ...,$$

we obtain the estimate

$$|f^{(n)}(a)| = \frac{n!}{2\pi}|\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta| \le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}2\pi R = \frac{M_R n!}{R^n}.$$

6.8 Morera Theorem

The inverse to Cauchy's theorem is Morera theorem. This theorem we shall state and prove below.

Theorem 6.4 *If $f(z)$ is a continuous function in a domain D, and if the integral*

$$\int_C f(z)\, dz = 0$$

along any closed curve C in D, then $f(z)$ is an analytic function.

Proof. Let a be a complex number in the domain D. Then, the function given by the formula

$$F(z) = \int_a^z f(\zeta\, d\zeta, \qquad z \in D$$

is independent of a path in D from a to z. Indeed, if C_1 and C_2 are two paths from a to z, then the curve $C = C_1 \cup -C_2$ is closed, and by the assumption, the integral

$$\int_{C_1 \cup -C_2} f(\zeta)\, d\zeta = \int_{C_1} f(\zeta)\, d\zeta - \int_{C_2} f(\zeta)\, d\zeta = 0.$$

Hence, we obtain

$$\int_{C_1} f(\zeta)\, d\zeta = \int_{C_2} f(\zeta)\, d\zeta.$$

Thus, $F(z)$ is an antiderivative of $f(z)$, so that

$$F'(z) = f(z), \qquad z \in D.$$

This means that $F(z)$ is an analytic function in D and therefore $F(z)$ has the second derivative $F''(z) = f'(z), \quad z \in D$. So that, $f(z)$ possesses first derivative and it is also analytic function in D.

6.9 Exercises

Question 6.1 *Show that for any integer m and n the integral*

$$\int_0^{2\pi} e^{imt} e^{-int}\, dt = \begin{cases} 2\pi, & m = n, \\ 0, & m \neq n \end{cases}$$

Question 6.2 *Evaluate the integral*

$$\int_C z\, \overline{z}\, dz,$$

along the positively oriented circle $|z| = 1$.

Question 6.3 *Evaluate the integral*

$$\int_C (x + y^2 + ixy)\, dz,$$

along the path

$$C: \quad z(t) = x(t) + iy(t) = \begin{cases} t + 2i, & 1 \le t \le 2, \\ 2 + i(4 - t) & 2 < t \le 3 \end{cases}$$

Question 6.4 *Show that if* C *is the circle* $z(t) = z_0 + re^{it}, \ 0 \le t < 2\pi$, *positively oriented then*

$$\int_C f(z)\, dz = ir \int_0^{2\pi} f(z_0 + re^{it}) e^{it}\, dt,$$

for any continuous function $f(z)$.

Question 6.5 *Let* $f(z)$ *be an analytic function inside and on the circle* C : $z(\theta) = a + re^{i\theta}$, $0 \le \theta \le 2\pi$. *show that*

$$f^{(n)}(a) = \frac{n!}{2\pi r^n} \int_0^{2\pi} e^{-in\theta} f(a + re^{i\theta}) d\theta,$$

for $n - 0, 1, 2, ...,$ *and for* z *interior to* C.

Question 6.6 *Evaluate the integral*

$$\int_C (z + \frac{1}{z})\, dz,$$

along the path

1. *(a)*

   $$z(t) = 2 + e^{it}, \qquad 0 \le t < 2\pi,$$

   (b)

   $$z(t) = 1 + i + 2e^{it}, \qquad 0 \le t < 2\pi,$$

   (c)

   $$z(t) = 3\ e^{it}, \qquad 0 \le t < 2\pi.$$

Question 6.7 *Evaluate the integral*

1. *(a)*

   $$\int_C [x + 3y^2 + i(y - 3x^2)] dz,$$

   along the path

   $$C: \quad z(t) = x(t) + iy(t) = \begin{cases} (1 + i)t^2, & 0 \le t \le 1, \\ (2 - t^2)i + 1 & 1 < t \le 2 \end{cases}$$

Question 6.8 *Use Cauchy's theorem to show that the integral*

$$\int_C z \ \sin z^2 \, dz$$

is independent of a path with end points $a = 1 - i$ *and* $b = 2 + i$*. Evaluate the integral.*

Question 6.9 *Consider the following function:*

$$F(z) = \int_C \frac{d\zeta}{\zeta - z}, \qquad C: \ |z - a| = R,$$

Show that

$$F(z) = \begin{cases} 2\pi \, i, & z \in D, \\ 0, & z \notin D \cup C, \end{cases}$$

where D *is the disk* $|z - a| < R$*.*

Question 6.10 *Use Cauchy Integral Formulas to evaluate the integral*

1. *(a)*

$$\int_{|z+2i|=2} \frac{dz}{z^2 - z + 1 - i},$$

(b)

$$\int_{|z+2i|=2} \frac{dz}{(z^2 + z + 1 - i)^3},$$

(c)

$$\int_{|z-1|=2} \frac{z^2 \, dz}{(4 - z^2)},$$

(d)

$$\int_C \frac{e^{-\frac{\pi z}{2}} \, dz}{z^2 + 4}, \qquad z(t) = 3i + 2e^{it}, \quad 0 \le t < 2\pi.$$

(e)

$$\int_C \frac{e^{-\frac{\pi z}{2}} \, dz}{(z^2 + 4)^3}, \qquad z(t) = 3i + 2e^{it}, \quad 0 \le t < 2\pi.$$

(f)

$$\int_C \frac{dz}{(4z + 1)^2}, \qquad C: \ |z - i| = 1.$$

(g)

$$\int_C \frac{\sin 4z + \cos 2z}{(z - 1)(z - 2)}, \qquad C: \ |z| = 3.$$

Question 6.11 *Use Cauchy inequality to obtain an estimate for the derivatives of* $f(z) = \sin z$.

Question 6.12 *Evaluate*

$$\int_{|z|=5} \frac{zf'(z)}{f(z)} dz,$$

where $f(z) = z^2 + z + 1$.

Chapter 7

Series

7.1 Power Series

Power series about a point $z = z_0$ is an infinite series of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \tag{7.1}$$

where $a_0, a_1, ..., a_n, ...,$ are complex valued coefficients.
In most cases, we shall consider a power series about $z = 0$, that is, the series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n=0}^{\infty} a_n z^n, \tag{7.2}$$

Example 7.1 *Let us consider the geometric series*

$$1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n, \tag{7.3}$$

The geometric series is convergent for $|z| < 1$, and diverges outside of the unit circle, so that

$$\sum_{n=0}^{\infty} z^n = \begin{cases} \dfrac{1}{1-z}, & |z| < 1, \\ \textit{diverges}, & |z| \geq 1. \end{cases}$$

Indeed, the partial sum

$$S_n = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad z \neq 1.$$

Clearly, we have

$$\lim_{n\to\infty} z^n = 0, \qquad \text{and} \qquad \lim_{n\to 0} S_n = \frac{1}{1-z}, \qquad \text{for} \quad |z| < 1.$$

For $|z| > 1$, the geometric series diverges because the necessary condition of convergence is not satisfied, that is

$$\lim_{n \to 0} z^n = \infty, \qquad |z| > 1.$$

Also, the geometric series diverges for $|z| = 1$, since then

$$z^n = \cos n\theta + i \sin n\theta$$

But $\sin n\theta$ or $\cos n\theta$ do not tent to zero when $n \to 0$. Thus, the necessary condition of convergence is not satisfied, too.
The following theorem on convergence of power series holds:

Theorem 7.1 *If the power series*

$$\sum_{n=0}^{\infty} a_n z^n,$$

converges at point $z = z_1,\ z_1 \neq 0$, *then it absolutely converges in the disc* $|z| < r = |z_1|$.
Also, if the power series diverges at the point $z = z_1$, *then it diverges outside of the circle, that is for* $|z| > r$.

Proof. Let us assume that the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

is convergent at point $z = z_1$, so that, the number series

$$\sum_{n=0}^{\infty} a_n z_1^n,$$

is convergent. This means that all terms $a_n z_1^n,\ n = 0, 1, ...,$ are bounded, so that, there exists a constant M for which

$$|a_n z_1^n| \leq M, \qquad n = 0, 1, ...,$$

Let us note that

$$|a_n z^n| = |a_n z_1^n|\ |\frac{z}{z_1}|^n \leq M|\frac{z}{z_1}|^n = Mq^n,$$

for $q = |\frac{z}{z_1}| < 1$.

By the comparison test, the series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent within of the circle $|z| = r = |z_1|$.
Now, let the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

diverges at point $z = z_1$. Then, the inequality $|a_n z^n| > |a_n z_1^n|$, holds for $|z| > r, \ \ n = 0, 1, ...,$
By the comparison test, again, we conclude that the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

is divergent for $|z| > r$.
Radius of Convergence. The radius R of the circle $|z| = R$ is called radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

if this series converges for $|z| < R$ and diverges for $|z| > R$.
We compute radius of convergence of a power series using the following formula:

$$R = \frac{1}{\lambda}, \qquad \lambda = \lim_{n\to\infty} \sup_{n} \sqrt[n]{|a_n|}.$$

The radius $R = \infty$ if $\lambda = 0$, and the radius $R = 0$ if $\lambda = \infty$.
Also, one can compute the radius of convergence of a power series using the following limit:

$$\lambda = \lim_{n\to\infty} \sup_{n} |\frac{a_{n+1}}{a_n}|.$$

Example 7.2 *To compute the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n + 1},$$

we find

$$a_n = \frac{1}{3^n + 1}, \qquad \lambda = \lim_{n\to\infty} \sqrt[n]{\frac{1}{3^n + 1}} = \frac{1}{3}, \qquad R = \frac{1}{\lambda} = 3.$$

Let us note that the series

$$\sum_{n=0}^{\infty} a_n z^n,$$

and the series of derivatives

$$\sum_{n=1}^{\infty} n a_n z^{n-1},$$

have the same radius of convergence, since then

$$\lambda = \lim_{n\to\infty} \sup_{n} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sup_{n} \sqrt[n]{n|a_n|}.$$

7.2 Taylor Series.

Let $f(z)$ be an analytic function at a complex number z_0. Then, the Taylor series of $f(z)$ about z_0 is defined by the formula

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

In the case when $z_0 = 0$, the Taylor series is called Maclaurin series.
The following theorem holds:

Theorem 7.2 *If $f(z)$ is an analytic function at z_0, then its Taylor series converges and*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

for $|z-z_0| < R$, where R is the radius of the circle within which the function $f(z)$ is analytic.

Proof. Let $f(z)$ be an analytic function within the circle $|z-z_0| = R$.
Then, by Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=R} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in D = \{z: \ |z-z_0| < R\}.$$

Let us note that

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} \\
&= \frac{1}{\zeta - z_0}[1 + \frac{z-z_0}{\zeta-z_0} + (\frac{z-z_0}{\zeta-z_0})^2 + \cdots + (\frac{z-z_0}{\zeta-z_0})^n + \cdots]
\end{aligned}$$

for $|\zeta - z_0| = R$.
Hence, we obtain

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}, \qquad |z-z_0| < |\zeta - z_0| < R.$$

Now, coming back to Cauchy formula, we find

$$\begin{aligned}
f(z) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=R} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_{|\zeta-z_0|=R} f(\zeta)\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\,d\zeta \\
&= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}\frac{n!}{2\pi i}\int_{|\zeta-z_0|=R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta
\end{aligned}$$

Hence, by the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|\zeta - z_0|=R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta,$$

we obtain the following Taylor series representation of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

for $|z - z_0| < R$.

Example 7.3 *It is easily to establish Taylor series of the following elementary functions:*

1. *(a)*

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \qquad z \in Z,$$

(b)

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!} + \cdots, \qquad z \in Z,$$

(c)

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{2n!} + \cdots, \qquad z \in Z,$$

(d)

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots + (-1)^{n+1}\frac{z^n}{n} + \cdots, \qquad |z| < 1.$$

Taylor series with n-th terms of a function $f(z)$ about a point a can be obtained in `Mathematica` by the command

```
Series[f[z],{z,a,n}]
```

For example, the output of the following commands:

```
f[z_]:=Sin[z]Exp[z];
Normal[Series[f[z],{z,0,6}]]
```

is the series

$$z + \frac{z^2}{1} + \frac{z^3}{3} - \frac{z^5}{30} - \frac{z^6}{90}.$$

7.3 Laurent Series

Let

$$D(z_0, r_1, r_2) = \{z \in Z: \ \ 0 < r_1 < |z - z_0| < r_2\}.$$

be the annulus with radii r_1 and r_2 and the center at the point z_0. A Laurent series is considered in the annulus $D(z_0, r_1, r_2)$. A Laurent series takes the following form:

$$\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

Let us note that a Laurent series consists of the principal part

$$\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n},$$

and the regular part

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

A Laurent series is said to be convergent in annulus $D(z_0, r_1, r_2)$ if both series in regular part and principal part are convergent, otherwise the Laurent series is divergent. Clearly, the series in the regular part is convergent in the disc

$$|z - z_0| < r_2, \qquad r_2 = \frac{1}{\lambda}, \qquad \lambda = \lim_{n \to \infty} \sup_n \sqrt[n]{|a_n|}.$$

and divergent outside of the disc, that is, for $|z - z_0| > r_2$.
The series in the principal part is divergent in the disc $|z - z_0| < r_1$, and convergent outside of the disc, that is for $|z - z_0| > r_1$. Thus, both parts of the Laurent series are convergent in the annulus $D(z_0, r_1, r_2)$.
The following theorem holds:

Theorem 7.3 *If $f(z)$ is an analytic function in the annulus $D(z_0, r_1, r_2)$, then the Laurent series of $f(z)$ is convergent and*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad z \in D(z_0, r_1, r_2),$$

where the coefficients

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \qquad n = 0 \pm 1, \pm 2, ...,$$

Proof. Let z be a complex number in the annulus $D(z_0, r_1, r_2)$ which boundary consists with two circles $C_1(z_0, r_1)$ and $C_2(z_0, r_2)$. Dividing the annulus in two parts as on figure 8.1, we obtain the domain D_1 and D_2 with the boundaries ∂D_1 and ∂D_2. By Cauchy Integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in D_2,$$

and

$$\frac{1}{2\pi i}\int_{\partial D_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0, \qquad z \in D_2,$$

Adding both integrals and canceling the integrals along the common parts of the boundaries ∂D_1 and ∂D_2, we obtain the formula

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in D(z_0, r_1, r_2). \tag{7.4}$$

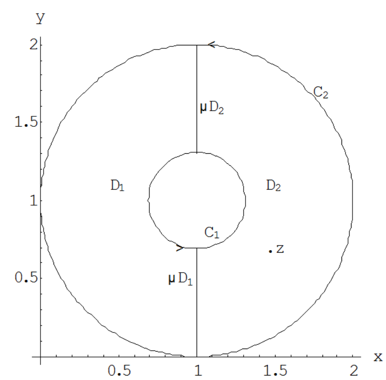

Fig. 8.1 Singular point z

Let us note that $|\zeta - z_0| > |z - z_0|$ for $\zeta \in C_2$, so that

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)}\\ &= \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}},\end{aligned}$$

Also, for $|\zeta - z_0| < |z - z_0|$, when $\zeta \in C_1$, we have

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)}\\ &= -\frac{1}{z - z_0}\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=1}^{\infty} \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n}.\end{aligned}$$

Both the above series are uniformly convergent with respect to ζ. Therefore, we can replace the function $\dfrac{1}{\zeta - z}$ in the integrals in (7.4) by these series.

Then, we obtain the following formula:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \; (z - z_0)^n \\ &+ \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \; (z - z_0)^{-n}. \end{aligned} \tag{7.5}$$

We can replace the two integrals in (7.5) along the circles C_1 and C_2 by the integral along a circle C within the annulus $D(z_0, r_1, r_2)$. Then, we obtain the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \qquad z \in D(z_0, r_1, r_2),$$

where the coefficients

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \qquad n = 0, \pm 1, \pm 2, ...,$$

End of the proof.

Example 7.4 *Let us consider the following function:*

$$f(z) = \frac{1}{z-2} + \frac{1}{z-1}.$$

This function is analytic in the annulus $D(0,1,2) = \{z \in Z : \; 1 < |z| < 2.\}$ Then

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \qquad |z| < 2,$$

and

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \sum_{n=1}^{\infty} \frac{1}{z^n}, \qquad |z| > 1.$$

Hence, the Laurent series of $f(z)$ is

$$f(z) = \frac{1}{z-2} + \frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \qquad 1 < |z| < 2.$$

A Laurent series with n terms of a function $f(z)$ about a point a can be easily obtained in `Mathematica` by the command

```
Series[f[z],{z,a,n}]
```

For example, let $f(z) = \dfrac{z}{z^2+1}$, $n = 6$, $a = i$. Then, we obtain, in `Mathematica`, six terms of the Laurent series by the following commands:

```
f[z_]:=z/(z^2+1);
Normal[Series[f[z],{z,I,6}]]
```

$$-\frac{I}{4}+\frac{1}{2(z-I)}+\frac{1}{8}(z-I)+\frac{1}{16}I(z-I)^2$$
$$-\frac{1}{32}(z-I)^3-\frac{1}{64}I(z-I)^4+\frac{1}{128}(z-I)^5+\frac{1}{256}I(z-I)^6.$$

7.4 Exercises

Question 7.1 *Find the region of convergence of the series*

1. (a)
$$\sum_{n=0}^{\infty}\frac{(z+2)^n}{(n+2)^3 4^{n+1}},$$
   (b)
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}.$$

Question 7.2 *Find the region of convergence of the series*

1. (a)
$$\sum_{n=1}^{\infty}\frac{1}{n^2+z^2},$$
   (b)
$$\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}.$$

Question 7.3 *Use the geometric series*

$$z+z^2+\cdots+z^n+\cdots=\frac{z}{1-z},\qquad |z|<1,$$

to show the following:

1. (a)
$$\sum_{n=1}^{\infty}r^n\cos n\theta=\frac{r\cos\theta-r^2}{1-2r\cos\theta+r^2},$$
   (b)
$$\sum_{n=1}^{\infty}r^n\sin n\theta=\frac{r\sin\theta-r^2}{1-2r\cos\theta+r^2}.$$

Question 7.4 *Find the Taylor series for the following functions about the indicated points*

1. *(a)*
$$f(z) = \cos z, \qquad z_0 = \frac{\pi}{2},$$
(b)
$$g(z) = \sinh z, \qquad z_0 = \pi i.$$

Question 7.5 *Find the Laurent series for the function*
$$f(z) = \frac{1}{z^2(1-z)},$$

1. *(a) in the annulus*
$$0 < |z| < 1,$$
(b) in the annulus
$$1 < |z| < \infty.$$

Question 7.6 *Find the Laurent series for the function*
$$f(z) = \frac{1}{z^2} + \frac{1}{1-z} + \frac{1}{2-z},$$

1. *(a) in the annulus*
$$0 < |z| < 1,$$
(b) in the annulus
$$1 < |z| < 2,$$
(c) in the annulus
$$2 < |z| < \infty.$$

Question 7.7 *Find principal part of the Laurent series for the following functions:*

1. *(a)*
$$f(z) = \frac{1}{z^2 \sin z}, \qquad z_0 = 0,$$
(b)
$$f(z) = \frac{e^{iz}}{z^2 + 4}, \qquad z_0 = 2i,$$
(c)
$$f(z) = \frac{z - \sin z}{z^3}, \qquad |z| > 0.$$

æ

Chapter 8

Residues

8.1 Singular Points

There are three types of singular points of a function $f(z)$ at a point z_0.

1. (a) z_0 is a *removable singular point* of $f(z)$ if the Laurent series of $f(z)$ about z_0 reduces to the regular part.

   (b) z_0 is *pole of order* m if the principal part of the Laurent series about z_0 has m terms, that is, $\cdots a_{-m-2} = a_{-m-1} = 0$, and $a_{-m} \neq 0$.

   (c) z_0 is *essential singular point* of $f(z)$ if the principal part of the Laurent series has infinite number of terms.

Example 8.1 *Let us consider the following function:*

$$f(z) = \frac{z-i}{z^2+1},$$

with the singular points $z_0 = i$ and $z_0 = -i$.

This function has the following Laurent series about the point $z_0 = i$:

$$\begin{aligned} f(z) = \frac{1}{z+i} &= \frac{1}{(z-i)+2i} = \frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}} \\ &= \frac{1}{2i}[1 - \frac{z-i}{2i} + (\frac{z-i}{2i})^2 - \cdots + (-1)^n(\frac{z-i}{2i})^n + \cdots] \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n(\frac{z-i}{2i})^n. \end{aligned}$$

Since the principal part of the Laurent series of $f(z)$ has been reduced to zero ($a_{-1} = a_{-2} = \cdots = a_{-m} = \cdots = 0$), therefore $z_0 = i$ is a removable singular point of $f(z)$.

Example 8.2 *Let us consider the following function:*

$$f(z) = \frac{1}{z^2+1}.$$

with the singular points $z_0 = i$ and $z_0 = -i$.

This function has the following Laurent series about $z_0 = i$:

$$\begin{aligned} f(z) = \frac{1}{z^2+1} &= \frac{1}{(z-i)(z+i)} \\ &= \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i} \\ &= \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{(z-i)+2i} \\ &= \frac{1}{2i}\frac{1}{z-i} + \frac{1}{4}\frac{1}{1+\dfrac{z-i}{2i}} \\ &\frac{1}{2i}\frac{1}{z-i} + \sum_{n=1}^{\infty}(-1)^n\frac{(z-i)^{n-1}}{(2i)^{n+1}}. \end{aligned}$$

The principal part of the series consists of one term and $a_1 = \dfrac{1}{2i}$. Thus, $z_0 = i$ is the pole of order one.

Example 8.3 *let us consider the following function*

$$f(z) = e^{\frac{1}{z}}.$$

This function has the following Laurent series about $z_0 = 0$:

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n} + \cdots$$

Because the principal part of the Laurent series has infinite number of terms, therefore $z_0 = 0$ is the essential singular point of $f(z)$.

8.2 Residues

Let $f(z)$ be an analytic function in the annulus $0 < |z - z_0| < r$. Then, the coefficient a_{-1} in the Laurent series

$$f(z) = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

is called *residue* of $f(z)$. In symbols, we write $Res\ f(z)\ |_{z_0} = a_{-1}$.

Example 8.4 *The function*

$$f(z) = \frac{1}{z^2(z-3)^2} = \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots$$

has the pole of order two at $z_0 = 3$ and the residue $Res\ f(z)\ |_{z_0=3} = a_{-1} = -\dfrac{2}{27}$.

If $f(z)$ has the Laurent series

$$f(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0)^n, \tag{8.1}$$

then the residue

$$a_{-1} = Res f(z)|_{z_0} = \frac{1}{2\pi i}\int_C f(z)\, dz. \tag{8.2}$$

Indeed, integrating both sides of (8.1), we obtain

$$\int_C f(z)\, dz = \sum_{n=1}^{\infty} a_{-n}\int_C (z - z_0)^{-n}\, dz + \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n\, dz,$$

where C is a closed contour within the annulus $0 < |z - z_0| < r$.
Because $\dfrac{(z - z_0)^{n+1}}{n+1}$ is the antiderivative to $(z - z_0)^n$, when $n \neq -1$, therefore, the integrals

$$\int_C (z - z_0)^n\, dz = 0, \qquad \text{for} \quad n \neq -1.$$

But, for $n = -1$, we have

$$\int_C \frac{dz}{z - z_0} = 2\pi i.$$

Hence, we obtain the formula (8.2).

Example 8.5 *Compute the residue of the function*

$$f(z) = \frac{z}{1 + z^2},$$

at the singular points $z_0 = i$ and $z_0 = -i$.

Let $C(i, 1): \ |z - i| = 1$, be the circle with radius $r = 1$ at the center $z_0 = i$. Then, by the formula (8.2), we compute the residue

$$a_{-1} = Res\frac{z}{1 + z^2}|_{z_0=i} = \frac{1}{2\pi i}\int_C \frac{z}{1 + z^2}\, dz = \frac{1}{2}\frac{1}{2\pi i}\int_C [\frac{1}{z - i} + \frac{1}{z + i}]dz = \frac{1}{2}.$$

Also, we compute

$$a_{-1} = Res\frac{z}{1 + z^2}|_{z_0=-i} = \frac{1}{2\pi i}\int_C \frac{z}{1 + z^2}\, dz = \frac{1}{2\pi i}\frac{1}{2}\int_C [\frac{1}{z - i} + \frac{1}{z + i}]dz = \frac{1}{2}.$$

for $z_0 = -i$ and $C(-i, 1): \ |z + i| = 1$.
If $f(z)$ has a simple singular pole at z_0, that is a pole of order one, then the residue

$$Res\ f(z)|_{z_0} = \lim_{z \to z_0}(z - z_0)f(z). \tag{8.3}$$

Indeed, in this case, the Laurent series of $f(z)$ is:

$$f(z) = a_{-1}\frac{1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Hence , we obtain

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

and

$$\lim_{z\to z_0}(z-z_0)f(z) = Res\ f(z)\ |z_0 = a_{-1}.$$

Example 8.6 *The function*

$$f(z) = \frac{1-z^2}{z(1+z^2)},$$

has simple poles at $z = i,\ z = -i$ *and* $z = 0$.
By the formula (8.3), we compute

$$Res\ f(z)|_{z=i} = \lim_{z\to i}(z-i)\frac{1-z^2}{z(z-i)(z+i)} = -1,$$

$$Res\ f(z)|_{z=-i} = \lim_{z\to -i}(z+i)\frac{1-z^2}{z(z-i)(z+i)} = 1,$$

$$Res\ f(z)|_{z=0} = \lim_{z\to 0} z\frac{1-z^2}{z(z-i)(z+i)} = 1.$$

Let us assume that $f(z)$ is an analytic function in a neighborhood of z_0, and has a pole of order m at the point z_0. Then, the residue of $f(z)$ at z_0 is given by the formula

$$Resf(z)|_{z_0} = \lim_{z\to z_0}\frac{\Phi^{(m-1)}(z)}{(m-1)!}, \qquad (8.4)$$

where the function

$$\Phi(z) = (z-z_0)^m f(z)$$

has a removable singular point at z_0.
If we define

$$\Phi(z_0) = a_{-m}, \qquad a_{-m} \neq 0,$$

then $\Phi(z)$ is analytic function also at z_0 and has the following series representation

$$\Phi(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

Hence, we compute the residue

$$a_{-1} = Res\ f(z)|_{z_0} = \lim_{z\to z_0}\frac{\Phi^{(m-1)}(z_0)}{(m-1)!}.$$

One can easily obtain a residue in `Mathematica` by the following command:

```
Residue[f[z],{z,a}]
```

For example, let $f(z) = \dfrac{z}{z^2+4}$, and $a = 2i$. Then we compute the residue

```
f[z_]:=z/(z^2+4);
Residue[f[z],{z,2*i}]
```

equal to $\dfrac{1}{2}$.

8.3 Residue Theorem

The following residue theorem holds:

Theorem 8.1 *Let C be a closed contour positively oriented. Suppose that $f(z)$ is an analytic function within and on the contour C, except for a finite number of singular points $z_1, z_2, ..., z_m$ interior to C. If $A_1, A_2, ..., A_m$ are residues at those singular points, then*

$$\int_C f(z)\,dz = 2\pi i(A_1 + A_2 + \cdots + A_m).$$

Proof. Let $C_k(z_k, r_k)$ be the circle with the center at z_k and the sufficiently small radius $r_k,\ k = 1, 2, ..., m$, so that, C_k is within the region enclosed by the contour C. Then, the circles $C_k,\ k = 1, 2, ..., m$, together with the contour C form the boundary ∂D of the region D, in which $f(z)$ is analytic.

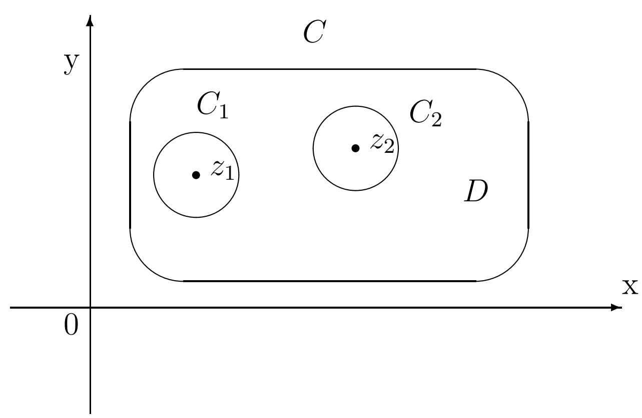

Fig. 8.2 Domain D

By Cauchy theorem

$$\int_{\partial D} f(z)\,dz = \int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz - \cdots - \int_{C_m} f(z)\,dz = 0.$$

Because

$$A_k = \frac{1}{2\pi i}\int_{C_k} f(z)\ dz, \qquad k = 1, 2, ..., m,$$

therefore

$$\int_C f(z)\ dz = 2\pi i(A_1 + A_2 + \cdots + A_m).$$

End of proof.

Example 8.7 *Let us evaluate the integral*

$$\int_{|z|=2} \frac{5z-4}{(z+1)(z-1)}\ dz.$$

The integrand has two singular points $z = 1$ and $z = -1$ both interior to the circle $C:\ |z| = 2$. The residues are:

$$Res f(z)|_{z=1} = \lim_{z\to 1}(z-1)\frac{5z-4}{(z-1)(z+1)} = \frac{1}{2},$$

$$Res f(z)|_{z=-1} = \lim_{z\to -1}(z+1)\frac{5z-4}{(z-1)(z+1)} = \frac{9}{2}.$$

By the residue theorem

$$\int_{|z|=2} \frac{5z-4}{(z+1)(z-1)}\ dz = 2\pi i(\frac{1}{2} + \frac{9}{2}) = 10\pi i.$$

Example 8.8 *Let us evaluate the integral*

$$\int_{|z|=2} \frac{z}{z^2+2z+2} dz,$$

using `Mathematica`.

The function $\frac{z}{z^2+2z+2}$, has two singular points $z_1 = -1+i$ and $z_2 = -1-i$ within the circle $|z| = 2$. We compute the residues at these points by the following commands:

```
f[z_]:=z/(z^2+2 z+2);
Residue[f[z],{z,-1+I};
Residue[f[z],{z,-1-I}];
```

equal to $A_1 = \frac{1}{2} + \frac{I}{2}$ and $A_2 = \frac{1}{2} - \frac{I}{2}$. By the residue theorem, the integral

$$\int_{|z|=2} \frac{z}{z^2+2z+2} dz = 2\pi i(A_1 + A_2) = 2\pi i,$$

8.4 Applications of the Residue Theorem

As an application of the residue theorem, we shall evaluate the following improper integral:

$$\int_{-\infty}^{\infty} f(x)\, dx.$$

The following theorem holds:

Theorem 8.2 *Let $f(z)$ be an analytic function in the upper half plane $Im\ z \geq 0$, except at the singular points $z_1, z_2, ..., z_m$ interior to the upper half of the complex plane, at which $f(z)$ has residue $A_1, A_2, ..., A_m$. If $f(z)$ is a real function on the x axis and satisfies the condition*

$$|f(z)| \leq \frac{M}{|R|^\alpha}, \qquad |z| \geq R,$$

for certain constants $M > 0,\ R > 0$ and $\alpha > 1$, then the infinite integral exists and its value is given by the formula

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i(A_1 + A_2 + \cdots + A_m).$$

Proof. Let C be the contour consisting with the interval $I = [-R, R]$ and the half of the circle $\Gamma :\ z(t) = Re^{it},\ 0 \leq t \leq \pi$, that is $C = I \cup \Gamma$,(see Figure 8.2). Let us choose R so large to be all singular points interior to the closed contour C.

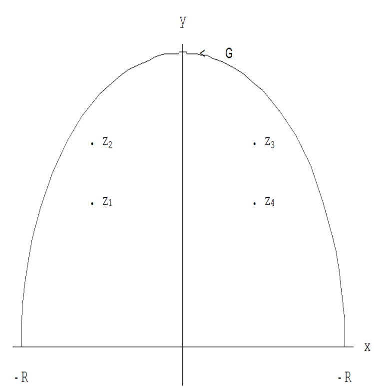

Fig. 8.2 Residue

By residue theorem

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{m} A_k.$$

By the assumptions, we have the following estimate of the integral along Γ:

$$|\int_\Gamma f(z)\, dz| \leq \frac{M}{R^\alpha}\pi R = \frac{\pi M}{R^{\alpha-1}}, \qquad \alpha > 1.$$

Clearly, if $R \to \infty$, then the integral along Γ tends to zero, and the first integral becomes

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{m} A_k.$$

End of the proof.

Example 8.9 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

We consider the function

$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$$

which has one singular point $z = i$ in the upper half of complex plane. By the residue theorem

$$\int_C \frac{dx}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_\Gamma \frac{dz}{1+z^2} = 2\pi\ i\ Res \frac{1}{1+z^2}|_{z=i} = \pi.$$

The function $f(z)$ satisfies the inequality

$$|f(z)| = \frac{1}{|1+z^2|} \leq \frac{1}{|z|^2-1} \leq \frac{1}{|z|^2 - \dfrac{|z|^2}{2}} = \frac{2}{|z|^2} \leq \frac{2}{R^2} \qquad \text{for} \quad |z| \geq R \geq 2.$$

Therefore, the integral

$$|\int_\Gamma \frac{dz}{1+z^2}| \leq \frac{1}{R^2}|\int_\Gamma dz| = \frac{\pi}{R}.$$

Hence, when $R \to \infty$, we obtain

$$\int_\Gamma \frac{dz}{1+z^2} = 0,$$

and

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

Example 8.10 *Show that*

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}, \qquad \Delta = b^2 - 4ac < 0, \quad a \neq 0. \tag{8.5}$$

We apply theorem 8.2. Let us consider the function

$$f(z) = \frac{1}{az^2+bz+c},$$

with real coefficients a, b and c and the discriminant $\Delta = b^2 - 4ac < 0$, $a \neq 0$. We shall show that the function $f(z)$ satisfies the assumptions of the theorem. Firstly, we note that $f(z)$ is the real valued function on x-axis, so that, for $z = x$,

$$f(x) = \frac{1}{ax^2 + bx + c}.$$

Secondly, the function $f(z)$ satisfies the inequality

$$|f(z)| \leq \frac{2}{|a|R^2}, \qquad |z| \geq R,$$

where $R = \frac{1}{|a|}[|b| + \sqrt{|b|^2 + 2|ac|}]$, $M = \frac{2}{|a|}$, $\alpha = 2$.
Indeed, to show the inequality, we estimate

$$|az^2 + bz + c| \geq |az^2| - |bz + c|$$

$$|bz + c| \leq |b||z| + |c| \leq \frac{1}{2}|a||z|^2,$$

$$\frac{1}{2}|a||z|^2 - |b||z| - |c| \geq 0,$$

The last quadratic inequality holds for $|z| \geq R = \frac{1}{|a|}[|b| + \sqrt{|b|^2 + 2|ac|}]$.
Hence, we obtain the estimate

$$\frac{1}{|az^2 + bz + c|} \leq \frac{1}{|az^2| - |bz + c|} \leq \frac{2}{|a||z|^2} \leq \frac{2}{|a|R^2}.$$

The function $f(z)$ has one singular point $z_1 = \frac{-b + i\sqrt{-\Delta}}{2a}$ in the upper half of complex plane at which

$$Resf(z)|_{z=z_1} = \lim_{z \to z_1}(z - z_1)\frac{1}{a(z - z_1)(z - \overline{z}_1)} = -\frac{i}{\sqrt{-\Delta}}.$$

By the theorem

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = 2\pi i Resf(z)_{|z=z_1} = \frac{2\pi}{\sqrt{4ac - b^2}}, \qquad a \neq 0.$$

Example 8.11 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 - x + 1}$$

We apply the formula (8.5), when $a = 1,\ b = -1,\ c = 1$, and the discriminant $\Delta = -3$.
Thus, we find

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 - x + 1} = \frac{2\pi}{\sqrt{3}}.$$

Example 8.12 *Let us evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}.$$

Clearly, the function

$$f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z-i)^{n+1}(z+i)^{n+1}}.$$

has a pole of order $n+1$ at $z_0 = i$ and $z_0 = -i$. Since, $z = i$ is only the singular point of $f(z)$ in upper half of complex plane, therefore, by the residue theorem

$$\int_C \frac{dz}{(1+z^2)^{n+1}} = 2\pi\ i\ Res\ f(z)|_{z=i}, \tag{8.6}$$

where the closed contour $C = [-R, R] \cup \Gamma$.
Let us compute the residue using formula (8.6). Clearly, we have $m = n+1$ and

$$\Phi(z) = (z-i)^{n+1}\frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z+i)^{n+1}}.$$

Because

$$Res\ f(z)|_i = \lim_{z\to i} \frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}} = \frac{(-1)^n}{(2i)^{2n+1}}\frac{(2n)!}{(n!)^2},$$

therefore, the integral

$$\begin{aligned}\int_C \frac{dz}{(1+z^2)^{n+1}} &= \int_{-R}^{R} \frac{dx}{(1+x^2)^{n+1}} + \int_\Gamma \frac{dz}{(1+z^2)^{n+1}} \\ &= 2\pi\ i\ Res\ f(z)|_{z=i} = \frac{(2n)!}{(n!)^2}\frac{\pi}{4^n}.\end{aligned}$$

Let us note that the function $f(z)$ satisfies the inequality

$$|f(z)| = \frac{1}{|(1+z^2)^{n+1}|} \leq \frac{2}{|z|^{2n+2}} \leq \frac{2}{R^{2n+2}}, \qquad |z| \geq R \geq 2.$$

Thus, we get the following estimate of the integral along the curve Γ:

$$|\int_\Gamma \frac{dz}{1+z^2}|^{n+1} \leq \frac{2\pi}{R^{2n+2}}.$$

Hence, when $R \to \infty$, we get

$$\int_\Gamma \frac{dz}{(1+z^2)} = 0,$$

and

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{(2n)!}{(n!)^2}\frac{\pi}{4^n}.$$

One can easily compute an improper integral in `Mathematica` using the following command:

$$\texttt{Integrate}[\texttt{f}[\texttt{x}], \{x, -\infty, \infty\}].$$

For example, we evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+8},$$

by the command:

$$\texttt{Integrate}[\frac{1}{x^2+8}, \{x, -\infty, \infty\}],$$

to get the value $\frac{\pi\sqrt{2}}{4}$.

Example 8.13 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx, \qquad a > 0.$$

Consider the function

$$f(z) = \frac{ze^{iz}}{z^2+a^2}$$

The fuction $f(z)$ has one simple singular point $z_1 = ia,\ a > 0$, in the upper half of the complex plane. By the residue theorem

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+a^2} dx + i \int_{\Gamma} \frac{ze^{iz}}{z^2+a^2} dx = 2\pi i A_1, \qquad a > 0.$$

where the residue

$$A_1 = \lim_{z->ia} (z-ia) \frac{ze^{iz}}{z^2+a^2} = \lim_{z->ia} \frac{ze^{iz}}{z+ia} = \frac{1}{2} e^{-a}$$

8.5 Exercises

Question 8.1 *Locate and classify the singularities of the following functions:*

1. (a)

$$f(z) = \frac{e^{iz}}{(z^2+z+1)^2},$$

   (b)

$$f(z) = \frac{z \sin z}{\cos z - 1}.$$

Question 8.2 *Find the residues at singular points for the following functions:*

1. (a)

$$f(z) = \frac{1+z}{z^2-2z},$$

(b)

$$f(z) = (z-3)\sin\frac{1}{z+2},$$

(c)

$$f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)},$$

(d)

$$f(z) = \frac{1}{\sin z},$$

(e)

$$f(z) = \frac{1}{(1+z^4)^2},$$

Question 8.3 *Using residue theorem, evaluate the following integrals:*

1. (a)

$$\int_{|z|=4} \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

(b)

$$\int_{|z|=1} \sin\frac{1}{z}dz,$$

(c)

$$\int_C \frac{\cosh z}{z^3}dz,$$

where C is the square with vertices $\pm 2 \pm 2i$. Ans. πi.

Question 8.4 *Using residue theorem evaluate the following infinite integrals*

1. (a)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2},$$

(b)

$$\int_0^{\infty} \frac{dx}{1+x^4},$$

(c)

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1},$$

(d)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx,$$

(e)

$$\int_{-\infty}^{\infty} \frac{\cos nx}{1 + x^2} dx,$$

(f)

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^8},$$

Question 8.5 *Use the function $f(z) = e^{-z^2}$, to evaluate Fresnel's integrals*

1. (a)

$$\int_{0}^{\infty} \sin x^2 \; dx,$$

(b)

$$\int_{0}^{\infty} \cos x^2 \; dx,$$

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