REAL ANALYSIS I

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To be used as supplementary text for **M302**, Real Analysis I, in the Department of Mathematics of the University of Botswana.

Krystyna STYŚ

Sergio SALBANY

Department of Mathematics
University of Botswana

Introduction

The purpose of these notes is to provide a standard text where the student can find statements and proofs of the relevant theorems. The notes cannot serve as a replacement for further reading and study, even though we have tried to arrange the material in an easily accessible and coherent manner.

The intention is to provide a basis for the justification of results and methods from the Calculus, as well as serve as a source for new concepts and developments.

The course is mainly concerned with real-valued functions and their special properties — continuity, differentiability, integrability are ever recurring themes throughout the course, and may be regarded as its essence or meaning, in a very broad sense.

Acknowledgement:

The authors wish to thank Prof T. Styś and Dr. D. Norton for their invaluable assistance.

Dr. Norton's comments, especially, laid bare many, many flaws in the original text. The ones that remain are the responsibility of the authors. They will find their way out in the course of time. Hopefully, they will not interfere with the understanding of the content.

The authors

To the Student

The first contact with formal rigour is, usually, not an easy one. Often, after a feeling of having mastered a section, you will return to it and realize that there is so much more to understand — well, that is part of learning and understanding, and it is only to be expected. Effort and perseverance cannot be overemphasized.

Contents

List of Figures

Chapter 1

Real Numbers

1.1 The Real Numbers as a Field

We are already familiar with basic properties of numbers that involve multiplication, addition, and other operations. We are going to express this knowledge in the form of axioms and then examine some consequences of the axioms.

First, we state the algebraic axioms for addition and multiplication.

Consider a set of objects F where there are two operations, called *addition* and *multiplication*, defined for any pair of objects in F. If a and b are two objects in F, then denote the image of (a, b) under the operations of *addition* and *multiplication* by $a + b$ and $a \bullet b$, respectively.

The set F, with the two operations of *addition* and *multiplication*, is called a **field**, if the following axioms **A1**–**A11** are satisfied.

Algebraic axioms of addition:

A1 *Closure property:*

For any two objects $a, b \in F$ there is one and only one object $a + b \in F$, called the **sum** of a and b.

A2 *Commutative law:*

For any two objects $a, b \in F$, $a + b = b + a$.

A3 *Associative law:*

For any three objects $a, b, c \in F$,

$$(a + b) + c = a + (b + c).$$

A4 *Existence of a* **zero**:

There is an object $0 \in F$, called **zero**, such that, for all a in F,

$$a + 0 = a.$$

A5 *Existence of an* **additive inverse**:

For each object a in F there is an object $x \in F$, such that

$$a + x = 0.$$

The number x is called the **additive inverse** of a and denoted by $-a$.

Algebraic axioms of multiplication:

A6 *Closure property*:

For any two objects $a, b \in F$, there is one and only one object $a \bullet b \in F$, called the **product** of a and b.

A7 *Commutative law*:

For any two objects $a, b \in F$, $a \bullet b = b \bullet a$.

A8 *Associative law:*

For any three objects $a, b, c \in F$, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

A9 *Existence of a* **unit**:

There is an object 1 in F, different from 0, such that for every object $a \in F$, $a \bullet 1 = a$.

A10 *Existence of a* **multiplicative inverse**:

For each object a in F different from 0 there is an object $x \in F$ such that

$$a \bullet x = 1.$$

This number x is called the inverse of a and is denoted by a^{-1}.

Distributive law:

A11 *Distributive law of multiplication with respect to addition:*

For any three objects $a, b, c \in F$,

$$a \bullet (b + c) = (a \bullet b) + (a \bullet c).$$

Example 1.1 *Checking the field axioms.*

Let $F_1 = \{x, y\}$ be a set of two distinct objects and let the addition and multiplication operations be defined by the following tables:

$+$	x	y
x	x	y
y	y	x

$\bullet$	x	y
x	x	x
y	x	y

so that

$$\begin{array}{ll} x + x = x & x \bullet x = x \\ x + y = y & x \bullet y = x \\ y + x = y & y \bullet x = x \\ y + y = x & y \bullet y = y \end{array}$$

Show that the set F_1 together with the operations $+$ and $\bullet$ defined above is a field.

Solution. Axioms **A1**, **A2**, **A6**, and **A7** are obvious for F_1. The verification of the axioms **A3**, **A8**, and **A11** are straightforward, case-by-case checks, which we now carry out. Note that the addition and multiplication tables provide a 0, namely x, and a 1, namely y, satisfying **A4** and **A9**.

Since there are only two objects in F_1, for each axiom, we need to check 8 equations obtained by setting a, b, c equal to x or y:

$$\begin{array}{lll} a = x, & b = x, & c = x \\ a = x, & b = x, & c = y \\ a = x, & b = y, & c = x \\ a = x, & b = y, & c = y \\ a = y, & b = x, & c = x \\ a = y, & b = x, & c = y \\ a = y, & b = y, & c = x \\ a = y, & b = y, & c = y \end{array}$$

$(a + b) + c$	$a + (b + c)$
$(x + x) + x = x$	$x + (x + x) = x$
$(x + x) + y = x + y$	$x + (x + y) = x + y$
$(x + y) + x = y + x = y$	$x + (y + x) = x + y = y$
$(x + y) + y = y + y = x$	$x + (y + y) = x + x = x$
$(y + x) + x = y + x$	$y + (x + x) = y + x$
$(y + x) + y = y + y$	$y + (x + y) = y + y$
$(y + y) + x = x + x = x$	$y + (y + x) = y + y = x$
$(y + y) + y = x + y = y$	$y + (y + y) = y + x = y$

$(a \bullet b) \bullet c$	$a \bullet (b \bullet c)$
$(x \bullet x) \bullet x = x \bullet x$	$x \bullet (x \bullet x) = x \bullet x$
$(x \bullet x) \bullet y = x \bullet y = x$	$x \bullet (x \bullet y) = x \bullet x = x$
$(x \bullet y) \bullet x = x \bullet x$	$x \bullet (y \bullet x) = x \bullet x$
$(x \bullet y) \bullet y = x \bullet y$	$x \bullet (y \bullet y) = x \bullet y$
$(y \bullet x) \bullet x = x \bullet x = x$	$y \bullet (x \bullet x) = y \bullet x = x$
$(y \bullet x) \bullet y = x \bullet y = x$	$y \bullet (x \bullet y) = y \bullet x = x$
$(y \bullet y) \bullet x = y \bullet x$	$y \bullet (y \bullet x) = y \bullet x$
$(y \bullet y) \bullet y = y \bullet y$	$y \bullet (y \bullet y) = y \bullet y$

Comment: All fields with two elements are of this type. Usually, the elements are written 0, 1 and the field is denoted by Z_2.

1.2 Consequences of the Axioms

P1 *The uniqueness of zero.*

Suppose that 0 and $0'$ are two elements of F such that, for every $a \in F$,

$$\text{(i)} \quad a + 0 = a,$$
$$\text{(ii)} \quad a + 0' = a.$$

Let $a = 0'$. Then $0' + 0 = 0'$, by (i).
Let $a = 0$. Then $0 + 0' = 0$, by (ii).
By **A2**, $0' + 0 = 0 + 0'$, which implies $0' = 0$.

P2 *The uniqueness of the additive inverse $-a$ of a.*

Given a, suppose there are two numbers x_1 and x_2 such that $a+x_1 = 0$ and $a+x_2 = 0$. Adding x_2 to both sides of the first equation and adding x_1 to both sides of the second equation gives

$$(a + x_1) + x_2 = x_2$$

and

$$(a + x_2) + x_1 = x_1.$$

Hence, by the commutative and associative laws, we get

$$a + (x_1 + x_2) = x_2$$

and

$$a + (x_1 + x_2) = x_1,$$

which implies that $x_1 = x_2$. This unique element will be denoted by $-a$. Thus $a + (-a) = 0$.

P3 For any two numbers a and b there exists a unique number x such that

$$a + x = b.$$

This number is given by $x = b + (-a)$.

To prove **P3** we shall show that

(i) x satisfies the equation $a + x = b$;
(ii) if x' also satisfies the equation, then $x' = x$.

Proof of (i). If $x = b + (-a)$, then by **A2**—**A4**,

$$a + x = a + [b + (-a)] = a + [(-a) + b] = [a + (-a)] + b = 0 + b = b + 0 = b.$$

Proof of (ii). Suppose that x' satisfies the equation $a + x' = b$. We shall show that $x = x'$.
Adding $(-a)$ to both sides of the equation $a + x' = b$, we get

$$(a + x') + (-a) = b + (-a) = x.$$

The left hand side is

$$(a+x')+(-a) = a+[x'+(-a)] = a+[(-a)+x'] = [a+(-a)]+x' = 0+x' = x'.$$

Thus we conclude

$$x' = b + (-a) = x.$$

P4 For $a \neq 0$ and any number b, there is a unique number x such that

$$ax = b.$$

This number will be denoted by $a^{-1}b$, or $\dfrac{b}{a}$.

Proof. We first verify that $x = a^{-1}b$ satisfies the equation $ax = b$:

$$ax = a(a^{-1} \cdot b) = (a \cdot a^{-1})b = 1 \cdot b = b.$$

Secondly, if x' satisfies $ax' = b$, then $a^{-1}(ax') = a^{-1}b$, so that $1 \cdot x' = a^{-1}b$. Hence $x' = a^{-1}b = x$.

P5 For all numbers a, $-(-a) = a$. If $a \neq 0$, then $(a^{-1})^{-1} = a$.

Proof. By **P3**, $-b$ is that unique number such that $b + (-b) = 0$. Now $(-a) + a = 0$. Thus, taking $b = -a$, we have $-(-a) = a$.

The proof of the other statement proceeds analogously.

P6 For all numbers a, we have $a \cdot 0 = 0$.

Proof. We have $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. Thus $b = b + b$, where $b = a \cdot 0$. Adding $-b$ to both sides, we get

$$(b + b) + (-b) = b + (-b).$$

Hence $b + (b + (-b)) = 0$ or $b + 0 = 0$, so that $b = 0$, as required.

1.3 The Real Numbers as an Ordered Field

It is known that the real numbers may be ordered in such a way that:

O1. $x \leq x$ (Reflexivity)

O2. $x \leq y$ and $y \leq x \implies x = y$ (Antisymmetry)

O3. $x \leq y$ and $y \leq z \implies x \leq z$ (Transitivity)

O4. For every x and y, one and only one of the following holds:
$x = y$ or $x < y$ or $y < x$. (Trichotomy)

As usual, $x \leq y$ denotes "x is smaller than y or equal to y", and $x < y$ stands for "x is smaller than y".

The order and the operations of addition and multiplication are related as follows:

OA. If $x \geq y$, then $x + z \geq y + z$, for all z.

OM. If $x \geq y$ and $z \geq 0$, then $x \cdot z \geq y \cdot z$.

Definition 1.1 *A field in which the above holds is called an* **ordered field***.*

Theorem 1.1 *In an ordered field,* $1 > 0$.

Proof. By trichotomy, either $1 > 0$ or $1 = 0$ or $1 < 0$, and only one of these relations holds. We know that $1 \neq 0$, so we may assume that $1 < 0$. Adding -1 to both sides, we get, by OA: $0 < -1$. Then $0 < (-1) \cdot (-1)$, by OM. Hence $0 < 1$, which contradicts $1 < 0$. Hence $0 < 1$, as required. ∎

Example 1.2 *Ordered Fields.*

(a) The set of rational numbers forms an ordered field.

(b) It is not possible to define an order relation in Z_2 in such a way that Z_2 becomes an ordered field. If it were possible, then $1 > 0$. Note that $1 = -1$, so that $-1 > 0$. Hence, by OA, $1 + (-1) > 0$. Thus $0 > 0$, contradicting the trichotomy property.

(c) It is not possible to define an order relation on the set of complex numbers in such a way that it becomes an ordered field. The justification will be left as an exercise. ∎

1.4 Order Completeness of the Reals

The representation of real numbers as points on a directed line illustrates an important feature of sets of real numbers —"those that are bounded above must have a smallest upper bound". The precise definitions follow.

Definition 1.2 .

(i) *Let $A \subseteq \mathbb{R}$. A is* **bounded above** *by M, equivalently, M is an* **upper bound** *of A if $a \leq M$ for all a in A.*

(ii) *M is said to be the* **least upper bound** *of A if M is an upper bound for A, and, if P is any other upper bound of A, then $M \leq P$.*

The definitions of lower bound and greatest lower bound are left as an exercise.

Definition 1.3 *An ordered set is said to be* **complete** *if every non-empty subset A which is bounded above has a least upper bound.*

We can now formulate the fundamental **Completeness Property of $\mathbb{R}$.**

Axiom of Completeness. $\mathbb{R}$ is a complete ordered field.

Let us examine some consequences of this axiom.

Theorem 1.2 Archimedean Property *For any real number a there is an integer n such that $a < n$.*

Proof. Suppose this were false. Then there would exist a real number a such that $n \leq a$ for all integers n. Thus, the set of integers would be bounded above. As such, it would have a least upper bound m (not necessarily an integer).

Thus $n + 1 \leq m$ for all integers n. Hence $n \leq m - 1$, for all n. Hence $m - 1$ would also be an upper bound for the set of integers. Hence $m \leq m - 1$ (since m is the **least** upper bound). This last inequality implies $m + 1 \leq m$, hence, adding $-m$ to both sides, $1 \leq 0$, which is impossible. This contradiction proves the theorem. ∎

Corollary 1.1 *Given $x > 0$, there is an integer n such that $x > 1/n > 0$.*
(Select n so that $n > 1/x$, noting that $x \neq 0$.)

Theorem 1.3 *Every nonempty subset B which is bounded below has a greatest lower bound m.*

Proof. The set $A = -B = \{-b \mid b \in B\}$ is bounded below. Hence A has a least upper bound ℓ. It is easily verified that $m = -\ell$ is the greatest lower bound of B. ∎

Theorem 1.4 Denseness Property of the Rational Numbers
Let $x < y$ be any real numbers. There are integers p, q such that

$$x < \frac{p}{q} < y.$$

Proof.
Case 1 $0 < x < y$.
By the Archimedean property, there exists q such that $q > 1/(y - x)$, which implies that $qy - qx > 1$ or

$$qy > 1 + qx. \tag{1.1}$$

Let $z = qx$. Consider the subset S_z of all natural numbers that are greater than z:

$$S_z = \{m \in \mathbb{N} \mid m > z\}.$$

By the Archimedean property, S_z is not empty. Denote by p the least element of S_z. Then we have the following inequality

$$p - 1 \leq z = qx < p,$$

which, combined with (1.1), gives

$$qx < p < qy \quad \text{or} \quad x < \frac{p}{q} < y.$$

Case 2 $x = 0$.
The case $x = 0$ is covered by the previous corollary.

Case 3 The case $x < 0 \leq y$ follows from $-y \leq 0 < -x$, and $x < y < 0$ follows from the case $0 < (-y) < (-x)$, discussed above. $\blacksquare$

Finally, not all real numbers are rational numbers.

Example 1.3

Let $A = \{x \in \mathbb{R} \mid x^2 < 2 \quad \text{or} \quad x < 0\}$. Then A is bounded above, by 2, for instance. Let m be the least upper bound of A. Since $5/4 \in A$, we have $1 < m$. Thus $m > 0$.

We show that $m^2 = 2$. If $m^2 < 2$, it is easy to verify that

$$0 < \frac{2 - m^2}{2(2m + 1)} \quad \text{and} \quad \left(m + \frac{2 - m^2}{2(2m + 1)}\right)^2 < 2,$$

which is impossible, by definition of m.

If $m^2 > 2$, then one can verify that

$$0 < \frac{m^2 - 2}{4m} \quad \text{and} \quad \left(m - \frac{m^2 - 2}{4m}\right)^2 > 2,$$

which again contradicts the definition of m. Thus $m^2 = 2$.

A familiar argument shows that there is no rational number whose square is 2.

1.5 Exercises

1.1 Consider the system with four elements $0, 1, 2, 3$ and the rules of *addition* and *multiplication* as given in the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\bullet$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	x	y
3	0	3	u	z

where $x, y, u, z \in \{0, 1, 2, 3\}$ and $x, y,$ and z are all different.

Is it possible to choose the values of x, y, u, z in such a way that the system is a field?

1.2 Find the least upper bound (supremum) and the greatest lower bound (infimum) of the following sets:

(i) All numbers of the form $2^{-n} + 3^{-m} + 5^{-p}$, where $n,\ m,\ p \geq 1$, are integers.

(ii) The set of all numbers x such that

$$x^2 + x - 1 < 0.$$

1.3 Let A and B be sets which are bounded above. Denote by $A+B$ the set of all numbers of the form $a + b$, where $a \in A$ and $b \in b$. Show that

$$\sup(A + B) = \sup A + \sup B.$$

If $A,\ B$ consist of positive numbers and $A \cdot B$ denotes all products of the form $a \cdot b$, where $a \in A,\ b \in B$, show that

$$\sup(A \cdot B) = (\sup A) \cdot (\sup B).$$

1.4 If $S \subseteq T$ and T is bounded above, show that S is bounded above and

$$\sup S \leq \sup T.$$

1.5 Let A be a non-empty subset of $\mathbb{R}$. Show that x is $\sup A$ if and only if x has the following properties:

(i) $a \leq x, \quad$ for all $\quad a \in A$;

(ii) for any $\varepsilon > 0$, there exists a in A, such that

$$x - \varepsilon < a \leq x.$$

1.6 Formulate an analogous characterization for $\inf A$.

1.7 Let $-A = \{-a \mid a \in A\}$. Show that, for $A \neq \emptyset$,

$$\sup(-A) = -\inf A,$$

$$\inf(-A) = -\sup A.$$

1.8 What is $\sup \emptyset$, $\inf \emptyset$, where $\emptyset \subset \mathbb{R}$ is the empty set?

1.9 $|x|$ is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Prove that the following hold for all x, y in $\mathbb{R}$.

(i) $|x| \geq 0$;

(ii) $|x| = 0 \iff x = 0$;

(iii) $|x|^2 = x^2$;

(iv) $|x| = \sqrt{x^2}$;

(v) $|x + y| \leq |x| + |y|$ (Triangle Inequality);

(vi) $|x + y| = |x| + |y| \iff xy \geq 0$.

1.10 Show that if $|x| \leq \varepsilon$, for all positive numbers ε, then $x = 0$.

1.11 In an ordered field, show that $x^2 \geq 0$ for every x.

1.12 Prove by Mathematical Induction, or otherwise, Bernoulli's inequality:

$$(1 + x)^n > 1 + nx,$$

provided $x > -1$, $x \neq 0$, n is an integer number greater than 1.

1.13 Let $x_1, x_2, \ldots, x_n$ be positive numbers. Prove the following Arithmetic-Geometric Means Inequalities. When does equality take place?

(i) $$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 \cdot x_2},$$

(ii) $$\frac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 \cdot x_2 \cdot x_3},$$

(iii) $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

Chapter 2

Sequences of Real Numbers

2.1 Introduction to Sequences

Definition 2.1 *A* **sequence** *of real numbers is a real-valued function f whose domain is the set of natural numbers N, i.e.*

$$f : \mathbb{N} \mapsto \mathbb{R}.$$

The function f which defines a sequence is a rule that assigns to each natural number n a unique real value, normally denoted by a_n:

$$f(n) = a_n, \quad n = 1, 2, \ldots$$

The number a_n is called the n-th **term** of the sequence and the corresponding sequence is denoted by the symbol $\{a_n\}$:

$$\{a_n\} = \{a_1,\ a_2,\ a_3,\ a_4,\ \ldots\}.$$

We will find it convenient to use the notations $\{b_n\}, \{c_n\}, \{d_n\}, \{x_n\}, \{y_n\}$, etc., in addition to $\{a_n\}$, to denote sequences, especially when we deal with two or more sequences at a time.

Example 2.1 *Defining sequences.*

(a) The function $f(n) = \dfrac{n}{n+1}, n \in \mathbb{N}$, defines a sequence with n-th term $a_n = \dfrac{n}{n+1}$, so that

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, \quad a_2 = \frac{2}{2+1} = \frac{2}{3}, \quad a_3 = \frac{3}{3+1} = \frac{3}{4},$$

and so on. We have

$$\{a_n\} = \{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots,\ \frac{n}{n+1},\ \ldots\}.$$

(b) The function $f(n) = \dfrac{(-1)^n}{n^2},\ n \in \mathbb{N}$, defines the sequence

$$\{\ b_n\ \} = \{-1,\ \frac{1}{4},\ -\frac{1}{9},\ \frac{1}{16},\ -\frac{1}{25},\ \frac{1}{36},\ \ldots\ \frac{(-1)^n}{n^2},\ \ldots\ \}.$$

(c) The function

$$f(n) = c_n = (-1)^n \frac{n^2}{n+2}, \quad n \in \mathbb{N},$$

defines the sequence

$$\{c_n\} = \{-\frac{1}{3}, \frac{4}{4}, -\frac{9}{5}, \frac{16}{6}, -\frac{25}{7}, \ldots, (-1)^n \frac{n^2}{n+2}, \ldots\}$$

(d) Consider a sequence with n-th term

$$d_n = \sin \frac{\pi n}{4}, \quad n = 1, 2, \ldots.$$

This sequence is defined by a periodic function

$$f(n) = \sin \frac{\pi n}{4}, \quad n = 1, 2, \ldots$$

with period $\omega = 8$, so that

$$f(n+8) = f(n) \quad \text{i.e.} \quad d_{n+8} = d_n, \quad n = 1, 2, \ldots.$$

The first 8 terms of the sequence are shown in the following table:

n	1	2	3	4	5	6	7	8
$\frac{\pi n}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$d_n = \sin(\frac{\pi n}{4})$	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0

Refer to Figure 2.1 which shows a graph for each of the four sequences considered above. Although, only the first 20 terms are shown for each sequence, we clearly see some differences in the behaviour of the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$, as n changes.

For example, $a_n = \dfrac{n}{n+1}$ is an increasing function of n, bounded by the numbers 0 and 1:

$$0 < a_n < 1,$$

and, as n gets larger, a_n gets closer to the number $a = 1$.

The terms of $\{b_n\}$ are positive for even values of n and negative for odd values of n. As n gets larger, b_n approaches the number 0.

Now, it is not hard to see that, as n increases,

$$c_n = (-1)^n \frac{n^2}{n+1}$$

does not get close to any number. Figure 2.2(a) shows more terms of $\{c_n\}$, and we can clearly see the trend: the absolute value $|c_n|$ of c_n increases without bound, as n increases.

Finally,

$$d_n = \sin \frac{\pi n}{4}$$

is a periodic, bounded function of n:

$$-1 \leq d_n \leq 1, \quad d_{n+8} = d_n, \quad n = 1, 2, \ldots$$

Refer to Figure 2.2(b), which shows the first 50 terms of the sequence, to have a clearer picture of the behaviour of the sequence.

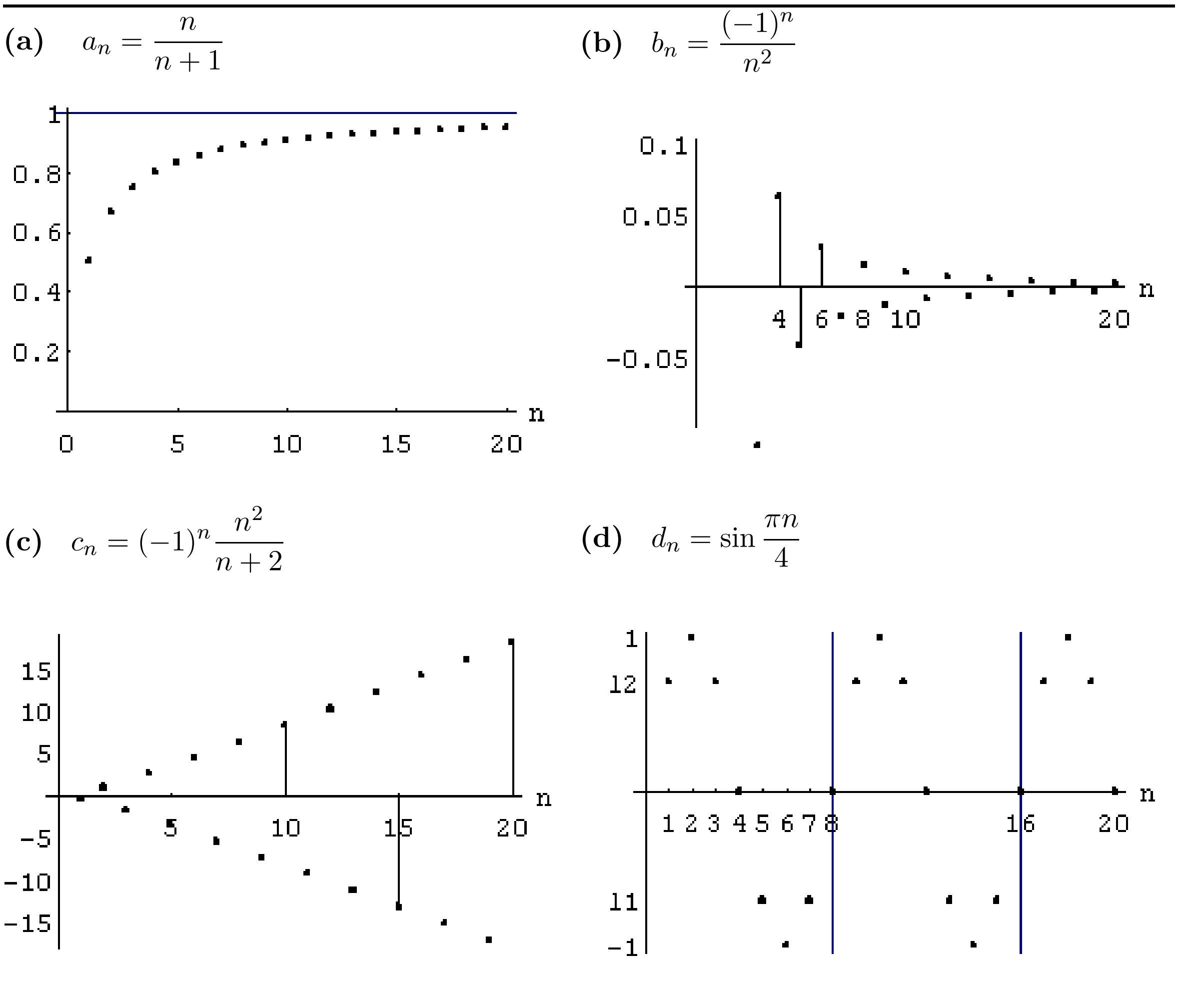


Figure 2.1: First terms of the sequences considered in Example 2.1.

(a) $c_n = (-1)^n \dfrac{n^2}{n+2}$ **(b)** $d_n = \sin \dfrac{\pi n}{4}$

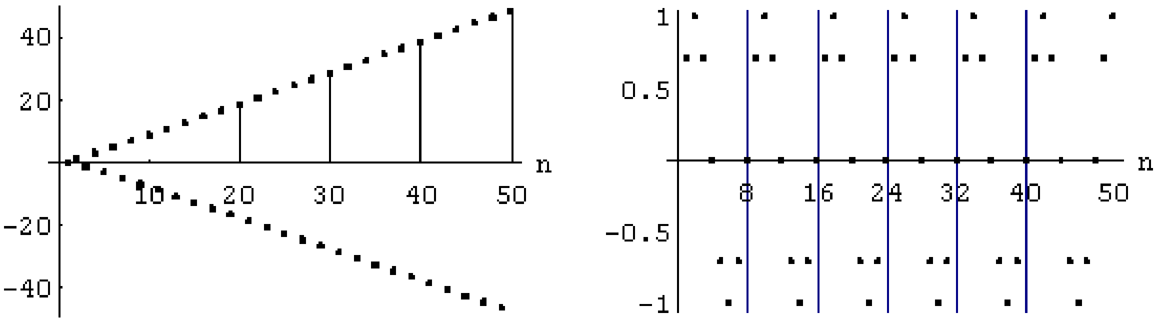


Figure 2.2: Examples of non-convergent sequences.

2.2 Definition of Convergence

Definition 2.2 *(Limit of a sequence)*
A sequence $\{a_n\}$ *is said to converge to the* **limit** a *if and only if for every* $\varepsilon > 0$ *there exists a natural number* N *such that*

$$|a_n - a| < \varepsilon \quad \textit{for} \quad n > N.$$

If $\{a_n\}$ *converges to* a *then we write* $\lim_{n\to\infty} a_n = a$.

Since

$$|a_n - a| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < a_n - a < \varepsilon \quad \Longleftrightarrow \quad a - \varepsilon < a_n < a + \varepsilon,$$

the definition requires that, given $\varepsilon > 0$, there is a number N (dependent on ε) such that all terms a_n of the sequence, for $n > N$, fall within the ε-neighbourhood of the limit a, that is within the interval $(a - \varepsilon, a + \varepsilon)$.

Using the quantifiers $\forall$ and $\exists$, the above definition can be written as follows.

$$\lim_{n\to\infty} a_n = a \quad \Longleftrightarrow \quad \forall\varepsilon > 0 \;\; \exists \; N \in \mathbb{N} \quad (n > N \quad \Longrightarrow \quad |a_n - a| < \varepsilon). \tag{2.1}$$

Intuitively, $\lim\limits_{n\to\infty} a_n = a$ if, as n increases, a_n gets **arbitrarily** close to a.

In light of the above, when we refer back to Figure 2.1, we can observe that the sequence $\{a_n\}$ converges to the limit $a = 1$ and $\{b_n\}$ converges to $b = 0$. The sequences $\{c_n\}$ and $\{d_n\}$ seem not to converge to any number.

Definition 2.3 *If the limit* $\lim_{n\to\infty} a_n$ *in* (2.1) *exists, we say that the sequence* $\{a_n\}$ **converges** *or is convergent. Otherwise, we say that the sequence* **diverges** *or is divergent.*

Theorem 2.1 *If* $\lim_{n\to\infty} a_n$ *exists, then it is unique.*

Proof. By definition 2.1 we have

$$\lim_{n\to\infty} a_n = a \quad \Longleftrightarrow \quad \forall \varepsilon_1 > 0 \ \exists\, N_1 \in \mathbb{N} \quad (n > N_1 \quad \Longrightarrow \quad |a_n - a| < \varepsilon_1)$$

and

$$\lim_{n\to\infty} a_n = a' \quad \Longleftrightarrow \quad \forall \varepsilon_2 > 0 \ \exists\, N_2 \in \mathbb{N} \quad (n > N_2 \quad \Longrightarrow \quad |a_n - a'| < \varepsilon_2).$$

Let ε be any positive number and let $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$. If $n > \max(N_1, N_2)$ then $|a_n - a| < \frac{\varepsilon}{2}$ and $|a_n - a'| < \frac{\varepsilon}{2}$. Consequently,

$$|a - a'| = |(a_n - a') - (a_n - a)| \le |a_n - a'| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies that $a = a'$, since $\varepsilon > 0$ is arbitrary. ∎

Example 2.2 *Showing that* $\lim_{n\to\infty} a_n = a$ *directly from the definition.*

(a) We shall show that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Let ε be any positive number. We have

$$|a_n - a| = |\frac{n}{n+1} - 1| = |-\frac{1}{n+1}| = \frac{1}{n+1} < \varepsilon,$$

provided $n + 1 > \frac{1}{\varepsilon}$ or $n > \frac{1}{\varepsilon} - 1$. Thus, if we choose N being the greatest integer number that is less than or equal to $\frac{1}{\varepsilon}$:

$$N = [\frac{1}{\varepsilon}],$$

then the condition

$$|\frac{n}{n+1} - 1| < \varepsilon$$

is satisfied for all $n > N$.

(b) We shall show that

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$$

Let $\varepsilon > 0$ be given. We have

$$|b_n - b| = |\frac{(-1)^n}{n^2} - 0| = \frac{1}{n^2} < \varepsilon \quad \text{provided} \quad n > N = \left[\frac{1}{\sqrt{\varepsilon}}\right]. \ ∎$$

It is clear that the value of N in (2.1) depends (normally) on ε, which is the case in our example:

$$N = \left[\frac{1}{\varepsilon} - 1\right] \quad \text{for the sequence} \quad \{a_n\} = \{\frac{n}{n+1}\}$$
$$N = \left[\frac{1}{\sqrt{\varepsilon}}\right] \quad \text{for the sequence} \quad \{b_n\} = \{\frac{(-1)^n}{n^2}\}.$$

Let us select some values of ε and see what are the corresponding values of N:

ε	0.1	0.05	0.02
$N = [\frac{1}{\varepsilon} - 1]$	9	19	49
$N = [\sqrt{\frac{1}{\varepsilon}}]$	3	4	7

In Figure 2.3, the sequences $\{a_n\}$ and $\{b_n\}$ are shown again, now with the number of terms increased to 50. In Figure 2.3(b), grid lines corresponding to the interval $(b-\varepsilon, b+\varepsilon)$ are included, for $\varepsilon = 0.02$.

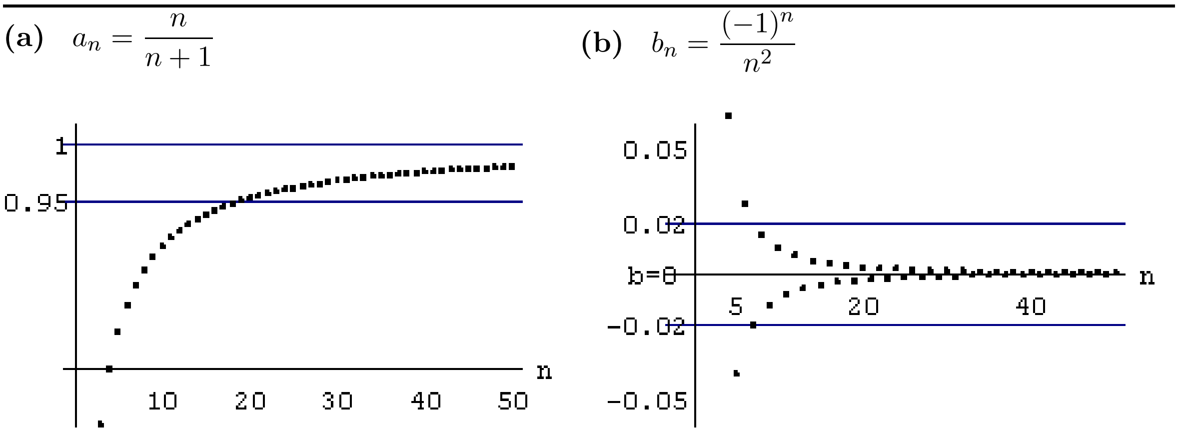


Figure 2.3: Illustrating convergence of the sequences $\{a_n\}$ and $\{b_n\}$ of Example 2.1.

Example 2.3 *Showing that* $\lim_{n\to\infty} a_n = a$ *directly from the definition — Pointing out that the smallest value of* N *in definition* 2.1 *is not necessarily needed.*

(a) $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

Let $\varepsilon > 0$ be given. We have

$$|a_n - a| = \left|\frac{n+1}{2n+3} - \frac{1}{2}\right| = \frac{1}{2(2n+3)} < \frac{1}{n}.$$

Now

$$n > N = \left[\frac{1}{\varepsilon}\right] \implies \frac{1}{n} < \varepsilon.$$

Hence, choosing $N = [\frac{1}{\varepsilon}]$, and assuming that $n > N$, we have

$$|a_n - a| < \frac{1}{n} < \varepsilon,$$

so that the inequality $|a_n - a| < \varepsilon$ holds for $n > N = [\frac{1}{\varepsilon}]$.

Note that we do not need to find the smallest possible value of N with the property that the inequality $|a_n - a| < \varepsilon$ is satisfied for all $n > N$; any value of N with this property is sufficient.

In fact, in this example it is not difficult to find the smallest possible value of N by solving the inequality

$$\frac{1}{2(2n+3)} < \varepsilon,$$

but, in general, this may be difficult.

(b) $\lim\limits_{n\to\infty} \dfrac{2n^3+1}{3n^3+n+4} = \dfrac{2}{3}.$

Let $\varepsilon > 0$ be given. We are to find a natural number N such that

$$|a_n - a| = \left|\frac{2n^3+1}{3n^3+n+4} - \frac{2}{3}\right| < \varepsilon \quad \text{for} \quad n > N. \tag{2.2}$$

We have

$$|a_n - a| = \left|\frac{6n^3+3-6n^3-2n-8}{3(3n^3+n+4)}\right| = \frac{|2n+5|}{3(3n^3+n+4)} < \frac{3n}{3n^3+n+4} < \frac{3n}{3n^3} = \frac{1}{n^2}$$

Now,

$$n > N = \left[\frac{1}{\sqrt{\varepsilon}}\right] \implies \frac{1}{n^2} < \varepsilon.$$

Hence, (2.2) holds with $N = \left[\dfrac{1}{\sqrt{\varepsilon}}\right]$.

(c) $\lim\limits_{n\to\infty} \dfrac{n+3}{n^2-5} = 0.$

Let $\varepsilon > 0$ be given. We have

$$|a_n - a| = \left|\frac{n+3}{n^2-5} - 0\right| = \left|\frac{n+3}{n^2-5}\right| = \frac{n+3}{n^2-5} \quad \text{for} \quad n > 2,$$

since $n^2 - 5 > 0$ for $n > 2$. Now,

$$\frac{n+3}{n^2-5} < \frac{n+n}{n^2-5} = \frac{2n}{n^2-5} < \frac{2n}{n^2-n^2/2} = \frac{4}{n} \quad \text{for} \quad n > 4,$$

and

$$\frac{4}{n} < \varepsilon \quad \text{for} \quad n > \frac{4}{\varepsilon}.$$

Thus

$$|a_n - a| < \frac{4}{n} < \varepsilon \quad \text{for} \quad n > N = \max\{2, 4, \left[\frac{4}{\varepsilon}\right]\} = \max\{4, \left[\frac{4}{\varepsilon}\right]\}.$$

(a) $\lim_{n\to\infty} r^n = 0, \ \ |r| < 1$ **(b)** $\lim_{n\to\infty} r^n = \infty, \ \ |r| > 1$

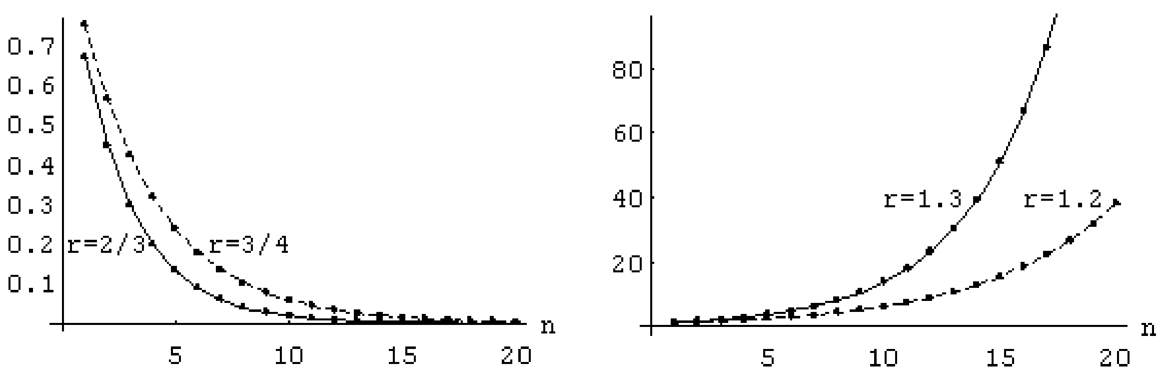


Figure 2.4: Illustrating convergence of r^n to 0 for $|r| < 1$ and divergence of r^n to ∞ for $|r| > 1$.

Example 2.4 *Showing that* $\lim_{n\to\infty} r^n = 0, \quad$ *if* $\quad |r| < 1.$

If $r = 0$ then $a_n = 0, \ \ n = 1, 2, \ldots$ and clearly $\lim_{n\to\infty} a_n = 0$. Suppose that $r \neq 0$. We are to find N such that

$$|a_n - a| = |r^n - 0| = |r|^n < \varepsilon \quad \text{for} \quad n > N. \tag{2.3}$$

Take $N = \dfrac{\log \varepsilon}{\log |r|}$. Since $|r| < 1$, $\log |r|$ is negative and we have

$$n > N = \tfrac{\log \varepsilon}{\log |r|} \quad \Longrightarrow \quad n \log |r| < \log \varepsilon$$

$$\Longrightarrow \quad \log |r|^n < \log \varepsilon.$$

Since $\log x$ is an increasing function,

$$\log |r|^n < \log \varepsilon \quad \Longrightarrow \quad |r|^n < \varepsilon.$$

Refer to Figure 2.4(a) which illustrates the convergence of r^n to 0, as $n \to \infty$, for $r = 2/3$ and $r = 3/4$. ∎

Definition 2.4 *(Infinite limit of a sequence)*

(a) *The sequence* $\{a_n\}$ *has* **limit** $+\infty$ *if for every positive number* M *there is a natural number* N *such that* $a_n > M$ *for all* $n > N$.

(b) *The sequence* $\{a_n\}$ *has* **limit** $-\infty$ *if for every positive number* M *there is a natural number* N *such that* $a_n < -M$ *for all* $n > N$.

If a_n has limit $+\infty$ then we write

$$\lim_{n\to\infty} a_n = +\infty.$$

If a_n has limit $-\infty$ then we write

$$\lim_{n\to\infty} a_n = -\infty.$$

Using the quantifiers $\forall$ and $\exists$, the above definition becomes:

(a)

$$\lim_{n\to\infty} a_n = +\infty \iff \forall M \ \exists\ N \in \mathbb{N} \quad (n > N \implies a_n > M). \tag{2.4}$$

(b)

$$\lim_{n\to\infty} a_n = -\infty \iff \forall M \ \exists\ N \in \mathbb{N} \quad (n > N \implies a_n < M). \tag{2.5}$$

Intuitively, $\lim_{n\to\infty} a_n = +\infty$ means that a_n increases without bound, as n increases, whereas $\lim_{n\to\infty} a_n = -\infty$ means that a_n decreases without bound, as n increases.

Thus, if $\lim_{n\to\infty} a_n = +\infty$ then with any positive value of M, no matter how large it is, we can find an N such that all terms a_n, for $n > N$, are greater than M. Similarly, if $\lim_{n\to\infty} a_n = -\infty$, then for any $M > 0$ there exists N such that all terms a_n, for $n > N$, are less than $-M$.

Example 2.5 *Infinite limits of sequences.*

(a)

$$\lim_{n\to\infty} \frac{n^2}{n+1} = +\infty.$$

Let M be any positive number. Following definition (2.4), we are to find N such that $a_n > M$ for all $n > N$. We have

$$a_n = \frac{n^2}{n+1} > \frac{n^2}{2n} = \frac{n}{2} > M, \quad \text{provided} \quad n > 2M,$$

so we can take $N = [2M]$.

(b)

$$\lim_{n\to\infty} r^n = +\infty, \quad \text{if} \quad r > 1.$$

Let $M > 0$ be given. Choose $N = [\frac{\log M}{\log r}]$. Then

$$n > N \implies n > \left[\frac{\log M}{\log r}\right] \implies n\log r > \log M$$

$$\implies \log r^n > \log M \implies r^n > M,$$

since $\log x$ is an increasing function.

Figure 2.4(b) illustrates the infinite limit of the sequence $\{r^n\}$, as $\to \infty$, for selected values of $r > 1$, namely $r = 1.2$ and $r = 1.3$.

(c)

$$\lim_{n\to\infty}\frac{-n^3}{n+1}=-\infty.$$

Let $M>0$ be given. We are to find N such that $a_n<-M$ for all $n>N$. We have

$$a_n=-\frac{n^2}{n+1}<-\frac{n^3}{2n}=-\frac{n^2}{2}<-M \quad \text{provided that} \quad n>\sqrt{2M}$$

so we can take $N=\left[\sqrt{2M}\right]$. ∎

2.3 Bounded Sequences

Theorem 2.2 *If* $\lim_{n\to\infty}a_n$ *exists then the sequence* $\{a_n\}$ *is bounded.*

Proof. Let $\lim_{n\to\infty}a_n=a$. Then

$$\forall\varepsilon>0 \;\; \exists\; N\in\mathbb{N} \quad (n>N \quad\Longrightarrow\quad a-\varepsilon<a_n<a+\varepsilon).$$

Let $\varepsilon=1$. There exists N such that

$$a-1<a_n<a+1, \quad \text{for} \quad n>N.$$

Therefore, it is clear that the sequence $\{a_n\}$ is bounded:

$$m\le a_n\le M, \quad n=1,2,\ldots,$$

where

$$m=\min(a_1,a_2,\ldots,a_N,a-1)$$

and

$$M=\max(a_1,a_2,\ldots,a_N,a+1). \;\blacksquare$$

2.4 The Algebra of Limits

It is clearly not always straightforward to use the definition of convergence to prove that a sequence $\{a_n\}$ converges to a known limit a. Moreover, if the limit a is not known, then the definition of convergence may not help in determining a.

Now we are going to introduce some useful results that enable us to evaluate limits of quite complicated sequences without appealing to the definition of convergence.

The following theorem can be used to evaluate the limits of sequences that arise by applying the arithmetic operations of addition, multiplication, and division on convergent sequences with known limits.

Theorem 2.3 *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences and let* c *be a real number. Then the sequences*

$$\{ca_n\}, \quad \{a_n+b_n\}, \quad \{a_nb_n\}$$

are convergent and the following rules apply.

(i) *Scalar product rule:*
$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$,

(ii) *Sum rule:*
$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

(iii) *Product rule:*
$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

(iv) *Quotient rule:*
If $b_n \neq 0$, *for* $n = 1, 2, \ldots$, *so that the sequence* $\{\frac{a_n}{b_n}\}$ *is defined, and if* $\lim_{n\to\infty} b_n \neq 0$, *then the sequence* $\{\frac{a_n}{b_n}\}$ *converges and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Proof. Let $\lim_{n\to\infty} a_n = a$ and let $\lim_{n\to\infty} b_n = b$. By Definition 2.2 we have

$$\lim_{n\to\infty} a_n = a \quad \Longleftrightarrow \quad \forall \varepsilon_1 > 0\ \exists\ N_1 \in \mathbb{N} \quad (n > N_1 \quad \Longrightarrow \quad |a_n - a| < \varepsilon_1) \tag{2.6}$$

and

$$\lim_{n\to\infty} b_n = b \quad \Longleftrightarrow \quad \forall \varepsilon_2 > 0\ \exists\ N_2 \in \mathbb{N} \quad (n > N_2 \quad \Longrightarrow \quad |b_n - b| < \varepsilon_2). \tag{2.7}$$

(i) Let $\varepsilon > 0$ be given and let $\varepsilon_1 = \frac{\varepsilon}{|c|+1}$. From (2.6) it follows that there exists N_1 such that

$$|a_n - a| < \varepsilon_1 \quad \text{for} \quad n > N_1.$$

Hence

$$|ca_n - ca| = |c||a_n - a| < |c|\varepsilon_1 = |c|\frac{\varepsilon}{|c|+1} \leq \varepsilon \quad \text{for} \quad n > N_1$$

which implies that $\lim_{n\to\infty} ca_n = ca = c \lim_{n\to\infty} a_n$.

(ii) Let $\varepsilon > 0$ be given and let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$. We have

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n > N = \max(N_1, N_2)$, which proves that

$$\lim_{n\to\infty} (a_n + b_n) = a + b = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(iii) Let $\varepsilon > 0$ be given. We are to find N such that

$$|a_n b_n - ab| < \varepsilon \quad \text{for} \quad n > N. \tag{2.8}$$

We have

$$\begin{aligned} |a_n b_n - ab| &= |(a_n b_n - ab_n) + (ab_n - ab)| \leq |(a_n b_n - ab_n)| + |(ab_n - ab)| \\ &= |a_n - a||b_n| + |b_n - b||a|. \end{aligned}$$

By theorem 2.2 we conclude that the convergent sequence $\{b_n\}$ is bounded and consequently the sequence $\{|b_n|\}$ is bounded too:

$$\exists M > 0 \quad |b_n| \le M \quad \text{for} \quad n = 1, 2, \dots.$$

If we choose $\varepsilon_1 = \dfrac{\varepsilon}{2M}$ in (2.6), then we have

$$|a_n - a||b_n| < \varepsilon_1 M = \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2} \quad \text{for} \quad n > N_1. \tag{2.9}$$

Now, by (2.7), with any $\varepsilon > 0$, $|b_n - b| < \varepsilon_2$ for $n > N_2$. Let

$$\varepsilon_2 = \frac{\varepsilon}{2} \frac{1}{1 + |a|}.$$

Then

$$|b_n - b||a| < \varepsilon_2 |a| = \frac{\varepsilon}{2} \frac{1}{1 + |a|} |a| = \frac{\varepsilon}{2} \frac{|a|}{1 + |a|} \le \frac{\varepsilon}{2}. \tag{2.10}$$

Finally, using (2.9) and(2.10) establishes the required result (2.8):

$$|a_n b_n - ab| \le |a_n - a||b_n| + |b_n - b||a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for} \quad n > N = \max(N_1, N_2).$$

(iv) Let ε be given. We are to show that there exists N such that

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|bb_n|} < \varepsilon \quad \text{for} \quad n > N. \tag{2.11}$$

By (2.7), using $\varepsilon_2 = \dfrac{b^2}{2}\varepsilon$, we conclude that there exists N_2 such that

$$|b_n - b| < \frac{b^2}{2}\varepsilon \quad \text{for} \quad n > N_2. \tag{2.12}$$

Since $\lim_{n\to\infty} b_n = b$, by **(i)** we conclude that $\lim_{n\to\infty} bb_n = b \lim_{n\to\infty} b_n = b^2$. Therefore

$$\forall \varepsilon_3 > 0 \ \exists N_3 \quad (n > N_3 \quad \Longrightarrow \quad |bb_n - b^2| < \varepsilon_3).$$

Let $\varepsilon_3 = b^2/2$ and let $n > N_3$. Then

$$\begin{aligned} |bb_n - b^2| < \varepsilon_3 \quad &\Longleftrightarrow \quad b^2 - \varepsilon_3 < bb_n < b^2 + \varepsilon_3 \\ &\Longrightarrow \quad b \cdot b_n > b^2 - \varepsilon_3 = b^2 - b^2/2 = b^2/2. \end{aligned}$$

Hence,

$$|bb_n| > \frac{b^2}{2} \quad \text{for} \quad n > N_3. \tag{2.13}$$

Using (2.12) and (2.13) we obtain

$$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|bb_n|} < \frac{b^2\varepsilon/2}{b^2/2} = \varepsilon \quad \text{for} \quad n > N = \max(N_2, N_3).$$

Thus, by the product rule,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n} = \lim_{n\to\infty} a_n \lim_{n\to\infty} \frac{1}{b_n} = a\frac{1}{b} = \frac{a}{b}. \ \blacksquare$$

2.5 The Squeeze Theorem for Sequences

When examining the convergence of a given sequence $\{a_n\}$, quite often it is possible to find two sequences, say $\{x_n\}$ and $\{y_n\}$, such that

$$x_n \leq a_n \leq y_n, \quad \text{for} \quad n > M \in \mathbb{N}\,,$$

so that, eventually, all terms of $\{a_n\}$ are "squeezed" between the corresponding terms of $\{x_n\}$ and $\{y_n\}$. If $\{x_n\}$ and $\{y_n\}$ converge to the same limit l, then the sequence $\{a_n\}$ must converge to the limit l.

Theorem 2.4 Squeeze theorem
Suppose that

$$x_n \leq a_n \leq y_n, \quad \textit{for} \quad n > M \in \mathbb{N} \tag{2.14}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l. \tag{2.15}$$

Then

$$\lim_{n\to\infty} a_n = l.$$

Proof. Let $\varepsilon > 0$ be given. We have to find a natural number N such that, for any $n > N$,

$$|a_n - l| < \varepsilon. \tag{2.16}$$

By assumption (2.15), we have

$$\exists N_1 \in \mathbb{N}\,(n > N_1 \implies l - \varepsilon < x_n < l + \varepsilon). \tag{2.17}$$

$$\exists N_2 \in \mathbb{N}\,(n > N_2 \implies l - \varepsilon < y_n < l + \varepsilon). \tag{2.18}$$

Let $N = \max(N_1, N_2, M)$. Then, if $n > N$, all the inequalities (2.17), (2.18), and (2.14) are true simultaneously. Thus, given any $\varepsilon > 0$, we have found a value of N such that

$$l - \varepsilon < x_n \leq a_n \leq y_n < l + \varepsilon,$$

which implies (2.16). ∎

Example 2.6 *Prove that* $\lim_{n\to\infty} r^n = 0$ *if* $|r| < 1$.

Solution. If $r = 0$ then $\{r^n\} = \{0\}$, hence $\lim_{n\to\infty} r^n = 0$. Let $r \neq 0$. Then

$$0 < |r| < 1 \implies \frac{1}{|r|} > 1 \implies \frac{1}{|r|} = 1 + d \quad \text{where} \quad d > 0.$$

Hence, by Bernoulli's inequality,

$$\frac{1}{|r|^n} = (1+d)^n > 1 + nd, \quad \text{for any} \quad n \geq 2,$$

which implies that

$$0 < |r|^n < \frac{1}{1+nd}, \quad \text{for} \quad n > 1.$$

Since $\lim_{n\to\infty} \dfrac{1}{1+nd} = 0$, by the squeeze theorem, we conclude that

$$\lim_{n\to\infty} r^n = 0. \quad \blacksquare$$

Example 2.7 *Prove that* $\lim_{n\to\infty} |r|^n = +\infty$ *if* $|r| > 1$.

Solution. If $|r| > 1$, then there exists a positive number d such that $|r| = 1 + d$. Thus, by the Bernoulli inequality,

$$|r|^n = (1+d)^n > 1 + nd \quad \text{for} \quad n \geq 2.$$

Now, let M be any positive number. Then

$$1 + nd > M \quad \text{provided that} \quad n > \frac{M-1}{d}.$$

Therefore, for any $M > 0$ there exists a natural number $N = \max\left(2, \dfrac{M-1}{d}\right)$ with the property that

$$|r|^n > 1 + nd > M \quad \text{for} \quad n > N.$$

This means that

$$\lim_{n\to\infty} |r|^n = +\infty. \quad \blacksquare$$

Figure 2.4(b) shows us the first twenty terms of the sequence $\{r^n\}$ for (a) $r = 1/4,\ r = 1/2$, and (b) $r = 3/2,\ r = 4/3$.

Example 2.8 *Prove that* $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$*, when x is any number.*

Solution. Let N be the smallest natural number such that $N > |x|$, so that $\alpha = |x|/N < 1$. Then, for $n \geq N$, we have

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{(N-1)!N(N+1)(N+2)\cdots n} \leq \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{n-N+1} = \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{-N+1} \alpha^n.$$

Hence

$$\frac{|x^n|}{n!} \leq c \cdot \alpha^n,$$

where c does not depend on n and $0 < \alpha < 1$. Since $\lim_{n\to\infty} \alpha^n = 0$ and

$$0 < \frac{|x^n|}{n!} \leq c\alpha^n, \quad \text{for} \quad n \geq N,$$

we have $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0. \quad \blacksquare$

Example 2.9 *Prove that* $\lim_{n\to\infty} \sqrt[n]{a} = 1,$ *for* $a > 0.$

Solution. We consider two cases.
Case 1. $a \geq 1.$

If $a \geq 1$ then $\sqrt[n]{a} \geq 1$ and $\sqrt[n]{a} = 1 + d_n$, where $d_n \geq 0$. Thus, by the Bernoulli inequality,

$$a = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \geq 2.$$

Since

$$\lim_{n\to\infty} \frac{a-1}{n} = 0,$$

by the Squeeze Theorem, it follows that

$$\lim_{n\to\infty} d_n = 0$$

and consequently

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 + \lim_{n\to\infty} d_n = 1.$$

Case 2. $0 < a < 1.$

If $0 < a < 1$, then $a = 1/b$, where $b > 0$, and

$$\lim_{n\to\infty} \sqrt[n]{b} = 1,$$

which implies

$$\lim_{n\to\infty} \sqrt[n]{a} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{b}} = 1. \quad \blacksquare$$

Example 2.10 *Prove that* $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Solution. We note that $\sqrt[n]{n} > 1$, when $n > 1$, so that we can write

$$\sqrt[n]{n} = 1 + d_n, \quad \text{where} \quad d_n > 0$$

and we have

$$n = (1 + d_n)^n = 1 + \binom{n}{1} d_n + \binom{n}{2} d_n^2 + \cdots + \binom{n}{n} d_n^n > \binom{n}{2} d_n^2.$$

Thus

$$n > \binom{n}{2} d_n^2 = \frac{n(n-1)}{2} d_n^2$$

which implies that

$$0 < d_n < \sqrt{\frac{2}{n-1}}, \quad \text{for} \quad n = 2, 3, \ldots,$$

Hence $\lim_{n\to\infty} d_n = 0$ and

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (1 + d_n) = 1. \quad \blacksquare$$

2.6 Monotone Sequences

Definition 2.5 *(Monotone sequence)*

(i) *The sequence* $\{a_n\}$ *is called* **increasing**, *if* $a_n \leq a_{n+1}$ *for all* $n \in \mathbb{N}$.

(ii) *The sequence* $\{a_n\}$ *is called* **strictly increasing**, *if* $a_n < a_{n+1}$ *for all* $n \in \mathbb{N}$.

(iii) *The sequence* $\{a_n\}$ *is called* **decreasing**, *if* $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$.

(iv) *The sequence* $\{a_n\}$ *is called* **strictly decreasing**, *if* $a_n > a_{n+1}$ *for all* $n \in \mathbb{N}$.

A sequence that is either increasing or decreasing is called a **monotone** *sequence.*

Example 2.11 *Monotone sequences.*

(a) $\{2^n\}$ is strictly increasing.
(b) $\{\frac{1}{3^n}\}$ is strictly decreasing.
(c) $\{(-1)^n n^2\}$ is not monotone. ∎

Theorem 2.5 Convergence of monotone sequences

(i) *If* $\{a_n\}$ *is increasing and bounded above, then it converges to its least upper bound:*

$$\lim_{n\to\infty} a_n = \sup a_n.$$

(ii) *If* $\{a_n\}$ *is decreasing and bounded below, then it converges to its greatest lower bound:*

$$\lim_{n\to\infty} a_n = \inf a_n.$$

Proof.

(i) We assume that $\{a_n\}$ is increasing and bounded above. Let $\sup a_n = M$. Then, given any $\varepsilon > 0$,

$$a_n \leq M \qquad \text{for all} \quad n \in \mathbb{N} ,$$

$$a_n > M - \varepsilon \qquad \text{for at least one value of } n.$$

Let this value of n be N_o. Now, $\{a_n\}$ is increasing, so that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and consequently $a_n \geq a_N > M - \varepsilon$ for $n > N_o$.

Hence, we have

$$M - \varepsilon < a_n < M + \varepsilon \quad \text{or equivalently} \quad |a_n - M| < \varepsilon \quad \text{for} \quad n > N_o.$$

Hence $\lim_{n\to\infty} a_n = M$.

(ii) If $\{a_n\}$ is decreasing and bounded below, then $\{b_n\} = \{-a_n\}$ is increasing and bounded above. Hence, by **(i)**, we conclude that $\{b_n\}$ converges and

$$\lim_{n\to\infty} b_n = \sup b_n.$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-b_n) = -\lim_{n\to\infty} b_n = -\sup b_n = \inf a_n. \quad ∎$$

Example 2.12

Let $d_i, \;\; i = 1, 2, \ldots$, be one of the decimal digits $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$. Consider the sequence $\{a_n\}$ defined as follows:

$$\begin{aligned} a_1 &= 0.d_1 \\ a_2 &= 0.d_1 d_2 \\ a_3 &= 0.d_1 d_2 d_3 \\ &\cdots\cdots\cdots\cdots \\ a_n &= 0.d_1 d_2 d_3 \cdots d_n \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

Since

$$a_{n+1} - a_n = 0.000 \cdots 0 d_{n+1} \geq 0,$$

the sequence is monotone increasing. Since $a_n < 1$, the sequence is bounded from above. Hence $\{a_n\}$ converges to a limit a (that is unique):

$$a = \lim_{n \to \infty} a_n.$$

This limit is the real number represented by the decimal expansion

$$a = 0.d_1 d_2 d_3 \cdots d_n \cdots \quad \blacksquare$$

(a) $\lim_{n \to \infty} x_n = \dfrac{1 + \sqrt{5}}{2} = l_1$ **(b)** $\lim_{n \to \infty} a_n = \sqrt{2} = l_2$

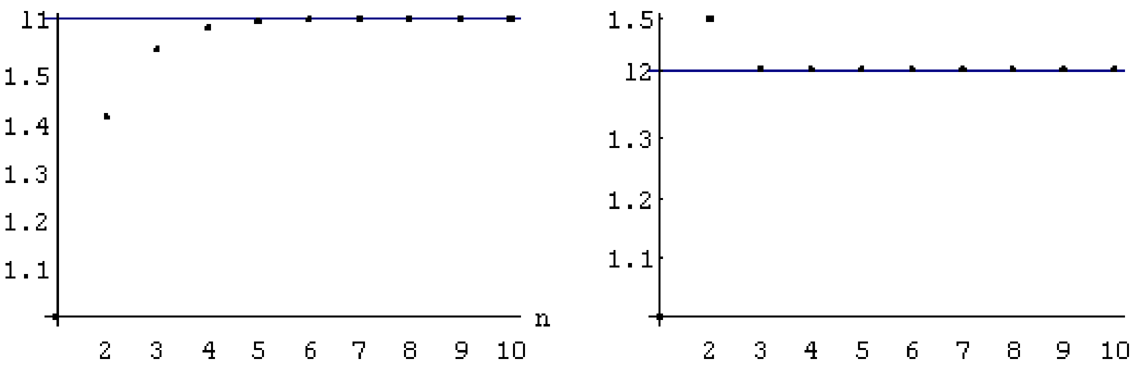


Figure 2.5: Illustrating convergence of the sequences $\{x_n\}$ and $\{a_n\}$ of Example 2.13

Example 2.13 *Examining convergence of sequences defined by a recurrence formula.*

(a) Let $\{x_n\}$ be given by

$$x_1 = 1, \quad x_{n+1} = \sqrt{1 + x_n}, \quad n = 1, 2, \ldots$$

The first ten terms of the sequence are shown in Figure 2.5(a). We shall show that

(i) $0 \le x_n \le 2$, for all n;

(ii) $x_{n+1} \ge x_n$, for all n;

(iii) $\lim_{n\to\infty} x_n = \dfrac{1+\sqrt{5}}{2}$.

(i) Clearly, $x_1 < 2$. Assume that $x_k < 2$. Prove that $x_{k+1} < 2$. We have

$$x_{k+1} = \sqrt{1 + x_k} < \sqrt{1+2} = \sqrt{3} < 2.$$

Thus, by Mathematical Induction, $x_n < 2$ for all n.

(ii) To prove that $x_{n+1} > x_n$, proceed by induction, once again. Clearly $x_2 > x_1$. Assume $x_{k+1} > x_k$. Then

$$x_{k+2} = \sqrt{1 + x_{k+1}} > \sqrt{1 + x_k} = x_{k+1}.$$

Hence $x_{n+1} > x_n$ for all $n \ge 1$.

(iii) By **(i)** and **(ii)**, $\{x_n\}$ is a monotone increasing sequence which is bounded above. As such it converges (to its least upper bound). Let the limit be l. Now

$$x_{n+1}^2 = 1 + x_n \;\; \& \;\; \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l \quad \Longrightarrow \quad l^2 = 1 + l.$$

Thus $l^2 - l - 1 = 0$, so that $l = (1 \pm \sqrt{5})/2$. But $x_n > 0$, $n \ge 1$, hence $l = \lim_{n\to\infty} x_n \ge 0$. Hence

$$l = \frac{1+\sqrt{5}}{2}. \;\blacksquare$$

(b) Let $\{a_n\}$ be given by

$$a_1 = 1, \quad a_{n+1} = \frac{a_n^2 + 2}{2a_n}, \quad n = 1, 2, \ldots$$

We shall show that $\{a_n\}$ converges and evaluate the limit.

Note that

$$a_1 = 1, \;\; a_2 = \frac{3}{2} = 1.5, \;\; a_3 = \frac{17}{12} = 1.4166\ldots, \;\; a_4 = \frac{577}{408} = 1.414215\ldots.$$

Refer to Figure 2.5(b) to see that $\{a_n\}$ converges fairly quickly to $\sqrt{2} = 1.41421356\ldots$ To prove that $\{a_n\}$ is a convergent sequence, we prove that **(i)** $\{a_n\}$ is decreasing for $n \ge 2$; **(ii)** $\{a_n\}$ is bounded below.

(i) First observe that $a_n > 0$, for $n \geq 1$, can be proved easily by induction. Then

$$\begin{aligned} a_{n+1} < a_n \quad &\Longleftrightarrow \quad \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) < a_n \\ &\Longleftrightarrow \quad a_n^2 + 2 < 2a_n^2 \\ &\Longleftrightarrow \quad 2 < a_n^2. \end{aligned}$$

Thus, we need to prove that $a_n^2 > 2$ for all $n \geq 2$. Now $a_2^2 > 2$. Assume $a_k^2 > 2$. We have

$$\begin{aligned} a_{k+1}^2 = \frac{1}{4}\left(a_k + \frac{2}{a_k}\right)^2 = \frac{1}{4}a_k^2 + 1 + \frac{1}{a_k^2} > 2 \quad &\Longleftrightarrow \quad \frac{1}{4}a_k^2 + \frac{1}{a_k^2} > 1 \\ &\Longleftrightarrow \quad a_k^4 + 4 > 4a_k^2 \\ &\Longleftrightarrow \quad a_k^4 - 4a_k^2 + 4 > 0 \\ &\Longleftrightarrow \quad (a_k^2 - 2)^2 > 0. \end{aligned}$$

Since the last inequality is true, it follows that $a_n^2 > 2$ for $n \geq 2$. Hence $a_{n+1} < a_n$ for $n \geq 2$, as required.

(ii) By **(i)**, we have $a_n \geq \sqrt{2}, \quad n \geq 2$.
Therefore we conclude that $\{a_n\}$ converges. Now we show that it must converge to $\sqrt{2}$.
Suppose $\lim_{n\to\infty} a_n = l$. Then we have

$$l = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{2}{\lim_{n\to\infty} a_n}\right) = \frac{1}{2}(l + \frac{2}{l}).$$

Hence

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) \quad \Longrightarrow \quad l^2 = 2 \quad \Longrightarrow \quad l = \pm\sqrt{2}.$$

Since $a_n \geq \sqrt{2}$ for all $n > 1$, l cannot be negative. Hence

$$l = \lim_{n\to\infty} a_n = \sqrt{2}. \quad \blacksquare$$

2.7 The Number e

One of the fundamental constants in Mathematical Analysis is the number

$$e = 2,7182818284\ldots.$$

It can be defined as a limit of an increasing sequence, an avenue which we now explore. Let

$$a_n = (1 + \frac{1}{n})^n, \quad n = 1, 2, \ldots$$

We shall show that **(i)**$\{a_n\}$ is strictly increasing; **(ii)** $\{a_n\}$ is bounded above. So that, by virtue of theorem 2.5, $\{a_n\}$ is convergent. Figure 2.6 illustrates the above concepts.

(a) $a_n = (1 + \dfrac{1}{n})^n, \ \ n = 1, 2, \ldots, 10.$ **(b)** $a_n = (1 + \dfrac{1}{n})^n, \ \ n = 11, 12, \ldots, 100.$

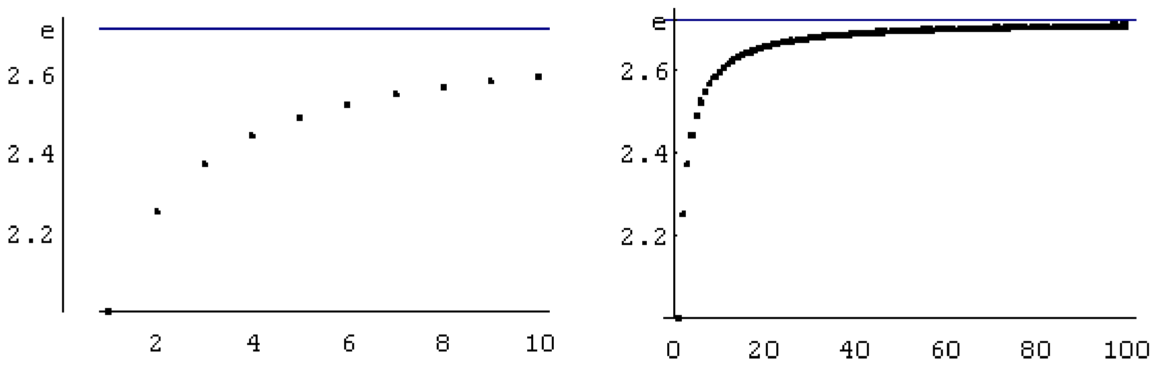


Figure 2.6: The first 100 terms of the sequence $a_n = (1 + \dfrac{1}{n})^n$ that converges to the number e.

(i) By the Bernoulli inequality,

$$(1 - \frac{1}{n^2})^n > 1 - \frac{1}{n} \quad \text{for} \quad n > 1.$$

We have

$$\begin{aligned} (1 - \frac{1}{n^2})^n > 1 - \frac{1}{n} \quad &\Longrightarrow \quad (1 + \frac{1}{n})^n (1 - \frac{1}{n})^n > 1 - \frac{1}{n} \\ &\Longrightarrow \quad (1 + \frac{1}{n})^n (1 - \frac{1}{n})^{n-1} > 1, \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$\left(1 + \frac{1}{n}\right)^n > \left(\frac{1}{1 - \frac{1}{n}}\right)^{n-1} = \left(\frac{n}{n-1}\right)^{n-1}$$

and we get

$$a_n = (1 + \frac{1}{n})^n > (\frac{n}{n-1})^{n-1} = a_{n-1}, \quad \text{for all} \quad n \in \mathbb{N} .$$

(ii) By the binomial expansion, for $n > 2$,

$$a_n = (1 + \frac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (\frac{1}{n})^k = 1 + 1 + \sum_{k=2}^{n} \binom{n}{k} (\frac{1}{n})^k = 2 + \sum_{k=2}^{n} \alpha_k.$$

The k-th term of the sum $\sum \alpha_k$ can be written as

$$\begin{aligned}\alpha_k = \binom{n}{k}(\frac{1}{n})^k &= \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{k!}(\frac{1}{n})^k \\ &= \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-k+2}{n}\frac{n-k+1}{n}\frac{1}{k!} \\ &= \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}),\end{aligned}$$

so that

$$\alpha_k < \frac{1}{k!} = \frac{1}{1\times 2\times 3\times\cdots\times k} < \frac{1}{1\times 2\times 2\times\cdots\times 2} = \frac{1}{2^{k-1}},$$

for $k = 2, 3, \ldots, n$. Therefore

$$\begin{aligned}a_n &= 1+1+\sum_{k=2}^{n}\alpha_k < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} \\ &= 1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1+2(1-(\frac{1}{2})^n) < 3,\end{aligned}$$

for all $n = 1, 2, \ldots$. ∎

2.8 Subsequences

Definition 2.6 *A sequence $\{b_k\}$ is called a* **subsequence** *of the sequence $\{a_n\}$ if there is a strictly increasing sequence of natural numbers $\{n_k\}$,*

$$n_1 < n_2 < \cdots < n_k < \cdots$$

such that

$$b_k = a_{n_k}, \quad k = 1, 2, \ldots$$

Example 2.14 *Subsequences and their limits.*

(a) Let $a_n = \left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}$.

Consider the subsequences $\{a_{2k}\}$, $\{a_{4k-1}\}$, and $\{a_{4k+1}\}$, $k = 1, 2, \ldots$ of the sequence $\{a_n\}$:

$$\begin{aligned}a_{2k} &= \left(1-\frac{1}{2k}\right)\sin\frac{2k\pi}{2} = \left(1-\frac{1}{2k}\right)\cdot 0 = 0, \quad k = 1, 2, \ldots \\ a_{4k-1} &= \left(1-\frac{1}{4k-1}\right)\sin\frac{(4k-1)\pi}{2} = \left(1-\frac{1}{4k-1}\right)\cdot(-1) = \frac{1}{4k-1}-1, \quad k = 1, 2, \ldots \\ a_{4k+1} &= 1-\frac{1}{4k+1}, \quad k = 1, 2, \ldots\end{aligned}$$

Note that, as $k \to \infty$, $a_{2k} \to 0$; $a_{4k-1} \to -1$; $a_{4k+1} \to 1$. Refer to Figure 2.7(a) to see clearly the three subsequences of $\{a_n\}$.

(b) Consider again the sequence $\{d_n\}$ of Example 2.1:

$$d_n = \sin \frac{\pi n}{4}.$$

Consider the following subsequences of $\{d_n\}$:

$$\begin{aligned}
d_{4k} &= \sin \frac{4k}{4}\pi = \sin \pi = 0, \quad k = 1, 2, \ldots \\
d_{8k-1} &= \sin \frac{8k-1}{4}\pi = -\sin \tfrac{\pi}{4} = -\tfrac{\sqrt{2}}{2}, \quad k = 1, 2, \ldots \\
d_{8k-2} &= \sin \frac{8k-2}{4}\pi = -\sin \tfrac{\pi}{2} = -1, \quad k = 1, 2, \ldots \\
d_{8k-3} &= \sin \frac{8k-3}{4} = -\sin \tfrac{3\pi}{4} = -\tfrac{\sqrt{2}}{2}, \quad k = 1, 2, \ldots \\
d_{8k+1} &= \sin \frac{8k+1}{4} = \sin \tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}, \quad k = 1, 2, \ldots \\
d_{8k+2} &= \sin \frac{8k+2}{4} = \sin \tfrac{\pi}{2} = 1, \quad k = 1, 2, \ldots \\
d_{8k+3} &= \sin \frac{8k+3}{4} = \sin \tfrac{3\pi}{4} = \tfrac{\sqrt{2}}{2}, \quad k = 1, 2, \ldots
\end{aligned}$$

(See Figure 2.7(b).) ∎

(a) $a_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$ **(b)** $d_n = \sin \frac{n\pi}{4}$, $l_1 = -\frac{\sqrt{2}}{2}$, $l_2 = \frac{\sqrt{2}}{2}$

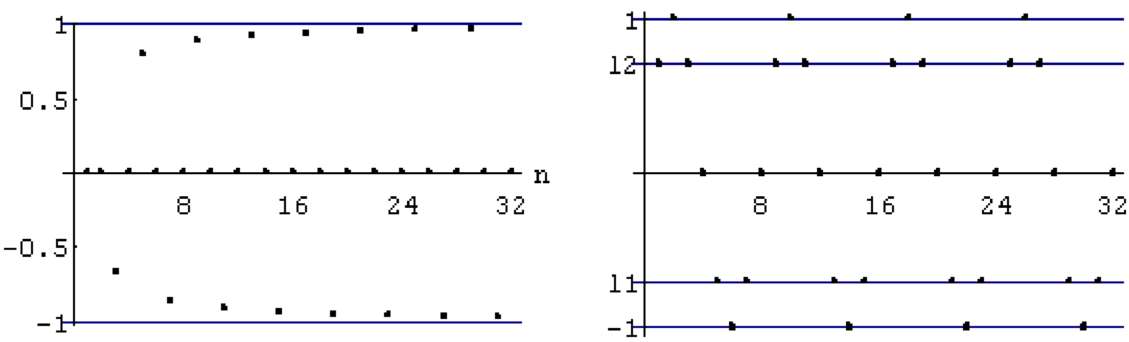


Figure 2.7: Subsequences.

Theorem 2.6 $\lim_{n\to\infty} a_n = A$ *if and only if every subsequence of* $\{a_n\}$ *converges to* A.

Proof.

$\Longrightarrow$ Assume that $\lim_{n\to\infty} a_n = A$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. To show that $\lim_{n\to\infty} a_{n_k} = A$, let $\varepsilon > 0$ be given. We know that there is N such that

$$n > N \quad \Longrightarrow \quad |a_n - A| < \varepsilon.$$

Now, $n_1 < n_2 < \cdots$, so there is K such that $N < n_K$, by the Archimedean property. Hence,

$$k > K \quad \Longrightarrow \quad n_k > n_K \quad \Longrightarrow \quad n_k > N \quad \Longrightarrow \quad |a_{n_k} - A| < \varepsilon,$$

as required.

$\Longleftarrow$ It is enough to observe that every sequence is a subsequence of itself. ∎

(a) $a_o = a_1 = 1, \;\; a_{n+1} = a_n + a_{n-1}$. **(b)** $r_n = \dfrac{a_n}{a_{n-1}}$.

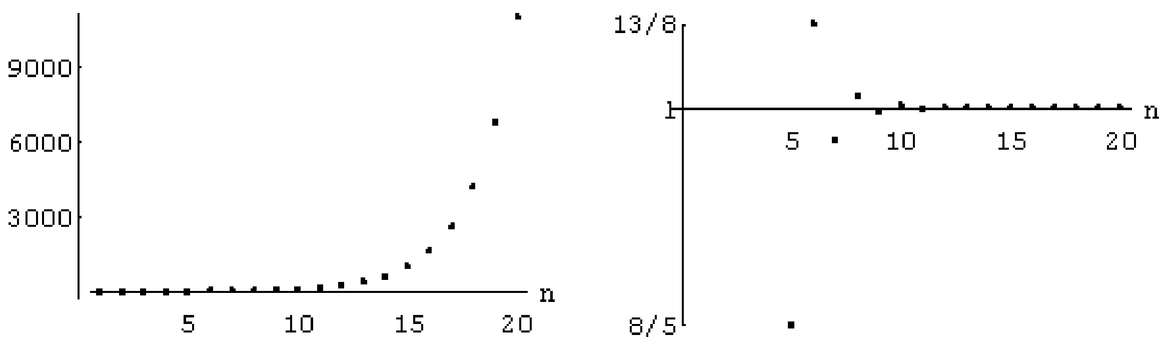


Figure 2.8: Fibonacci sequence $\{a_n\}$ and the sequence $\{r_n\}$ of ratios of the consecutive terms of $\{a_n\}$.

Example 2.15 *Show that the Fibonacci sequence* $\{a_n\}$ *defined by*

$$a_o = a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}, \;\; n = 1, 2, \ldots,$$

diverges.

Solution. We shall show that $\{a_n\}$ is unbounded above and hence diverges (see Figure 2.8(a)).

Note that

$$a_o = 1, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 5, \quad a_5 = 8, \quad a_6 = 13, \quad a_7 = 21, \; cdots.$$

It is natural to expect that $a_n \geq n$ for all n. This is certainly true for $n = 1, 2$. Assume that $a_r \geq r$ for all $r \leq k$, where $k \geq 2$. We shall show that $a_{k+1} \geq k + 1$, so that the result follows by Mathematical Induction. Since $k \geq 2$, we have

$$a_{k+1} = a_k + a_{k-1} \quad \Longrightarrow \quad a_{k+1} \geq k + (k-1) \geq 2 + (k-1) = k + 1. \;\; ∎$$

Example 2.16

Let $\{a_n\}$ be the Fibonacci sequence and consider the sequence $\{r_n\}$ of ratios of the consecutive terms of $\{a_n\}$:

$$r_n = \frac{a_n}{a_{n-1}}, n = 1, 2, \ldots$$

Show that the sequence $\{r_n\}$ converges and find its limit.

Solution. To prove that $\{r_n\}$ converges, we shall consider separately the two subsequences $\{r_{2k}\}$ and $\{r_{2k-1}\}$ of $\{r_n\}$ and show that they converge to the same limit:

$$\lim_{k\to\infty} r_{2k} = \lim_{k\to\infty} r_{2k-1}.$$

We list the first few terms of $\{r_n\}$:

$$1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \frac{13}{8}, \quad \frac{21}{13} \ldots$$

and claim that

$$1 \le r_n \le 2, \quad n = 1, 2, \ldots \tag{2.19}$$

We note that

$$r_{k+1} = \frac{a_{k+1}}{a_k} = \frac{a_k + a_{k-1}}{a_k} = 1 + \frac{a_{k-1}}{a_k} = 1 + \frac{1}{r_k}.$$

Hence, assuming that $1 \le r_k \le 2$, we conclude that

$$1 < 1 + \frac{1}{2} \le 1 + \frac{1}{r_k} = r_{k+1} \le 1 + \frac{1}{1} = 2,$$

and the claim (2.19) is proved by Mathematical Induction.

If $n > 3$, then

$$r_n = 1 + \frac{1}{r_{n-1}} = 1 + \frac{1}{1 + \frac{1}{r_{n-2}}} = 1 + \frac{r_{n-2}}{1 + r_{n-2}}. \tag{2.20}$$

Thus

$$r_{n+2} - r_n = \frac{r_n - r_{n-2}}{(1 + r_n)(1 + r_{n-2})},$$

which implies that $r_{n+2} - r_n$ and $r_n - r_{n-2}$ have both the same sign. Now

$$r_3 - r_1 = \frac{3}{2} - 1 > 0 \qquad \Longrightarrow \qquad r_{2k+1} - r_{2k-1} > 0, \;\; k = 1, 2, 3, \ldots$$

$$r_4 - r_2 = \frac{5}{3} - 2 < 0 \qquad \Longrightarrow \qquad r_{2k+2} - r_{2k} < 0, \;\; k = 1, 2, \ldots$$

Hence $\{r_{2k-1}\}$ is a monotone increasing sequence, bounded by 2 from above, and as such converges. Similarly we conclude that $\{r_{2k}\}$ converges as a monotone decreasing sequence that is bounded from below (by the number 1).

Let

$$l_1 = \lim_{k\to\infty} r_{2k-1}, \qquad l_2 = \lim_{k\to\infty} r_{2k}.$$

By (2.20), we have

$$l_1 = 1 + \frac{l_1}{1+l_1} \quad \text{and} \quad l_2 = 1 + \frac{l_2}{1+l_2},$$

so that both l_1 and l_2 satisfy the equation

$$l^2 - l - 1 = 0.$$

There are two solutions to the above equation: $(1 \pm \sqrt{5})/2$. By (2.19), $l_1 > 0$ and $l_2 > 0$. Hence

$$l_1 = l_2 = \lim_{n\to\infty} r_n = \frac{1+\sqrt{5}}{2}.$$

(See Figure 2.8(b)). ∎

2.9 Bolzano-Weierstrass Theorem

One of the fundamental properties of bounded sequences of real numbers is expressed by the following theorem.

Theorem 2.7 Bolzano-Weierstrass theorem

Every bounded sequence has a convergent subsequence.

Lemma 2.1 *Every sequence in* $\mathbb{R}$ *has a monotone subsequence.*

Proof. Given a sequence $\{a_n\}$ of real numbers, we must constuct a subsequence $\{a_{n_k}\}$ which is either increasing or decreasing.

Consider the set

$$S_N = \{a_{N+1}, a_{N+2}, \ldots\}$$

that, given N $(N = 1, 2, \ldots)$, contains all terms $\{a_n\}$ of the sequence $\{a_n\}$ for $n > N$.

Now, we distinguish two cases.

(i) Every set S_N, $N = 1, 2, \ldots$, has a largest element.

(ii) At least one of the sets $S_1, S_2, \ldots$ has no largest element.

In case **(i)**, we can construct a decreasing subsequence $\{a_{n_k}\}$ of $\{a_n\}$ as follows:

$$\begin{aligned}
a_{n_1} &= \max_{n>1} a_n = \max S_1,\\
a_{n_2} &= \max_{n>n_1} a_n = \max S_{n_1},\\
a_{n_3} &= \max_{n>n_2} a_n = \max S_{n_2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
a_{n_k} &= \max_{n>n_k} a_n = max S_{n_k}\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Obviously,

$$n_1 < n_2 < n_3 < \cdots$$

and

$$S_1 \supset S_{n_1} \supset S_{n_2} \supset \cdots,$$

which implies that

$$\max S_1 \geq \max S_{n_1} \geq \max S_{n_2} \geq \cdots$$

and the subsequence so constructed is decreasing:

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \cdots.$$

If we deal with case **(ii)**, then there exists a natural number M such that the set S_M does not have a largest element. Hence, for any a_m with $m > M$ there exists an a_n, such that $n > m$ and $a_n > a_m$. Then we can construct an increasing subsequence $\{a_{n_k}\} = \{c_k\}$ as follows.

Let $c_1 = a_{M+1}$ and let c_2 be the first term of $\{a_n\}$ following c_1 such that $c_2 > c_1$. Now, let c_3 be the first term of $\{a_n\}$ following c_2 for which $c_3 > c_2$, and so on. Hence

$$c_1 < c_2 < c_3 < \cdots. \quad \blacksquare$$

We now prove the Bolzano-Weierstrass Theorem.

Proof (of Bolzano-Weierstrass Theorem)
By the above lemma we conclude that the sequence $\{a_n\}$ has a monotone subsequence $\{a_{n_k}\}$. Since $\{a_n\}$ is bounded, so is its subsequence $\{a_{n_k}\}$. Thus $\{a_{n_k}\}$, $k = 1, 2, \ldots$ is a monotone sequence that is bounded and consequently $\{a_{n_k}\}$ converges, as $k \to \infty$. $\blacksquare$

2.10 Limit Superior and Limit Inferior

Definition 2.7 *A real number x is called a* **cluster point** *of the sequence $\{a_n\}$ if there exists a subsequence of $\{a_n\}$ that converges to x.*

Let C denote the set of all cluster points of a given sequence $\{a_n\}$. By the Bolzano-Weierstrass theorem, each bounded sequence has at least one convergent subsequence, and consequently at least one cluster point.

If $\{a_n\}$ is a convergent sequence with $\lim\limits_{n\to\infty} a_n = a$, then C consists of one point only, the limit of $\{a_n\}$:

$$C = \{a\}.$$

The set of all cluster points of a given sequence can be $\mathbb{R}$. The reader is asked to find an example of such a sequence.

Theorem 2.8 *Let $\{a_n\}$ be bounded and let C denote the set of all cluster points of $\{a_n\}$. Then C has a supremum and an infimum.*

Lemma 2.2 *Suppose that* $\lim_{n\to\infty} a_n = a.$

(i) *If* $a_n \geq m, \;\; n = 1, 2, \ldots,$ *then* $a \geq m.$

(ii) *If* $a_n \leq M, \;\; n = 1, 2, \ldots,$ *then* $a \leq M.$

Proof.
(i) We assume that $\lim_{n\to\infty} a_n = a$ and $a_n \geq m, \;\; n = 1, 2, \ldots$ Therefore

$$\forall \varepsilon > 0 \;\; \exists N \quad (n > N \quad \Longrightarrow \quad a - \varepsilon < a_n < a + \varepsilon).$$

Since $a_n \geq m, \;\; n = 1, 2, \ldots,$ we have $m \leq a_n < a+\varepsilon$ for $n > N$. This implies that $m < a+\varepsilon$ and the inequality holds with any value of $\varepsilon > 0$. Thus we conclude that

$$\lim_{n\to\infty} a_n = a \geq m.$$

(ii) We have

$$a_n \leq M \quad \Longrightarrow \quad -a_n \geq -M$$

and **(i)** is applicable. Thus $\lim_{n\to\infty} (-a_n) = -a \geq -M$ which implies $a \leq M$. ∎

Definition 2.8 *Let* $\{a_n\}$ *be a bounded sequence. Denote by* C *the set of cluster points of* $\{a_n\}$*. Then* C *is a nonempty bounded set. We define:*
(i) $\limsup_{n\to\infty} a_n = \overline{\lim}\, a_n$ *to be* $\sup C$*;*

(ii) $\liminf_{n\to\infty} a_n = \underline{\lim}\, a_n$ *to be* $\inf C$.

When $\{a_n\}$ *is unbounded above we define* $\limsup_{n\to\infty} a_n = \overline{\lim}\, a_n = +\infty.$

When $\{a_n\}$ *is unbounded below we define* $\liminf_{n\to\infty} a_n = \underline{\lim}\, a_n = -\infty.$

Example 2.17

Let $a_n = (-1)^n, \;\; n = 1, 2, \ldots$ Then $C = \{-1, 1\}$ and $\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1.$

Theorem 2.9 Existence of limits
Let $\{a_n\}$ *be a bounded sequence.* $\{a_n\}$ *converges if and only if*

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. The proof is left as an exercise.

2.11 Cauchy Sequences

If $\lim_{n\to\infty} a_n = a$, the definition of convergence implies that the terms a_n of the sequence $\{a_n\}$ get arbitrarily close to the limit a as n gets larger. In this situation the terms of the sequence must be getting closer to each other.

The above idea allows us to formulate a condition for convergence that does not explicitly involve the limit a of the sequence.

Definition 2.9 *A sequence $\{a_n\}$ is a* **Cauchy sequence** *if, for every $\varepsilon > 0$ there exists a natural number N such that*

$$|a_m - a_n| < \varepsilon \quad \text{for} \quad m, n > N. \tag{2.21}$$

Theorem 2.10 *Every convergent sequence is a Cauchy sequence.*

Proof. Let $\lim_{n\to\infty} a_n = a$. Then

$$\forall \varepsilon_1 > 0 \ \exists N \quad (n,\ m > N \quad \Longrightarrow \quad |a_n - a| < \varepsilon_1 \ \ \& \ \ |a_m - a| < \varepsilon_1). \tag{2.22}$$

We are to show that the Cauchy condition (2.21) is satisfied.

Let $\varepsilon > 0$ be given and let ε_1 in (2.22) be $\varepsilon_1 = \varepsilon/2$. We have

$$|a_m - a_n| = |(a_m - a) - (a_n - a)| \le |a_m - a| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

provided $n > N$ and $m > N$. But this shows that (2.21) holds. ∎

Example 2.18 *Verifying that a sequence is a Cauchy sequence.*

(a) We shall show that $\{a_n\} = \left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence.

Let $n > m$. We have

$$|a_m - a_n| = |\frac{m}{m+1} - \frac{n}{n+1}| = |\frac{m-n}{(m+1)(n+1)}| = |\frac{n-m}{n+1}\frac{1}{m+1}| \le \frac{1}{m+1} \le \frac{1}{m}.$$

Therefore

$$\forall \varepsilon > 0 \ \exists N = [\frac{1}{\varepsilon}] \quad (m, n > N \quad \Longrightarrow \quad |a_m - a_n| \le \frac{1}{m} < \varepsilon).$$

(b) We shall show that the sequence $\{b_n\}$ defined by the recursive formula

$$b_1 = \alpha, \quad b_2 = \beta, \quad b_{n+2} = \frac{1}{2}(b_{n+1} + b_n), \ \ n = 1, 2, \ldots$$

is a Cauchy sequence.

We have

$$\begin{aligned} |b_{n+2} - b_{n+1}| &= |\frac{1}{2}(b_{n+1} + b_n) - b_{n+1}| = \frac{1}{2}|(b_{n+1} - b_n)| \\ &= \frac{1}{2^2}|b_n - b_{n-1}| = \frac{1}{2^3}|b_{n-1} - b_{n-2}| = \cdots = \frac{1}{2^n}|\alpha - \beta|. \end{aligned}$$

Hence, if $n > m$,

$$\begin{aligned}|b_n - b_m| &= |(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \cdots + (b_{m+1} - b_m)| \\ &\le |b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| + \cdots + |b_{m+1} - b_m| \\ &\le \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-1}}\right)|\beta - \alpha| \\ &\le \frac{|\beta-\alpha|}{2^{m-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-m-1}}\right) \\ &= \frac{|\beta-\alpha|}{2^{m-1}}\frac{1-(\frac{1}{2})^{n-m}}{1-\frac{1}{2}} = \frac{|\beta-\alpha|}{2^{m-2}}\left(1-(1/2)^{n-m}\right) \\ &\le \frac{|\beta-\alpha|}{2^{m-2}}.\end{aligned}$$

Let $\varepsilon > 0$ be given. There exists N such that $\dfrac{|\beta-\alpha|}{2^{N-2}} < \varepsilon$. Then, for any $n > N$ and $m > N$,

$$|b_n - b_m| \le \frac{|\beta-\alpha|}{2^{N-2}} < \varepsilon.$$

Thus $\{b_n\}$ is a Cauchy sequence. ∎

Now we shall prove that the condition

$$\forall\varepsilon > 0 \ \exists N \in \mathbb{N} \quad (m, n > N \quad \Longrightarrow \quad |a_m - a_n| < \varepsilon). \tag{2.23}$$

which defines a Cauchy sequence is a sufficient condition for convergence of the sequence $\{a_n\}$.

Theorem 2.11 Cauchy condition of convergence

A sequence $\{a_n\}$ is convergent if $\{a_n\}$ is a Cauchy sequence.

Proof. We assume that the condition (2.23) is satisfied and prove that the sequence $\{a_n\}$ converges. Since (2.23) holds for any $\varepsilon > 0$, we can use $\varepsilon = 1$ to conclude that there exists N such that $|a_m - a_n| < 1$ for all $n > N$ and $m > N$. In particular,

$$|a_m - a_{N+1}| < 1 \quad \text{for all} \quad m > N,$$

which implies that

$$a_{N+1} - 1 < a_m < a_{N+1} + 1, \quad m > N.$$

Therefore the set

$$S = \{a_m \ | \ m > N\},$$

that contains all but a finite number of terms of $\{a_n\}$, is a bounded. This implies that the sequence $\{a_n\}$ is bounded.

Clearly,

$$M' \leq a_n \leq M, \quad \text{for all} \quad n \in \mathbb{N}\,,$$

where

$$M' = \min(a_1, a_2, \ldots, a_N, a_{N+1} - 1)$$

and

$$M = \max(a_1, a_2, \ldots, a_N, a_{N+1} + 1).$$

By the Bolzano-Weierstrass Theorem, $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$. Let

$$\lim_{n\to\infty} a_{n_k} = a.$$

Then

$$\forall \varepsilon_1 > 0 \;\; \exists N_1 \quad (n_k > N_1 \quad \Longrightarrow \quad |a_{n_k} - a| < \varepsilon_1).$$

It is assumed that $\{a_n\}$ is a Cauchy sequence, so that

$$\forall \varepsilon_2 > 0 \;\; \exists N_2 \quad (m, n > N_2 \quad \Longrightarrow \quad |a_m - a_n| < \varepsilon_2).$$

Let $N = \max(N_1, N_2)$. Let $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$. Then, for $n > N$ and $n_k > N$, we have

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies that $\{a_n\}$ converges and its limit is a. ∎

2.12 Infinite Series

Definition 2.10 *Let $\{a_k\}$, $k = 0, 1, 2, \ldots,$ be a given sequence of real numbers. Consider the sequence $\{S_n\}$ defined as the sum of the first $n+1$ terms of $\{a_k\}$:*

$$S_n = a_o + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k. \tag{2.24}$$

If $\{S_n\}$ converges to the limit S,

$$\lim_{n\to\infty} S_n = S,$$

then we define $\sum_{k=0}^{\infty} a_k$ to be S:

$$S = \sum_{k=0}^{\infty} a_k, \tag{2.25}$$

and call S the **sum of the infinite series** (2.25). *The series is then said to be* **convergent**; *otherwise it is said to be* **divergent**. *The sum S_n defined by* (2.24) *is called the* **n-th partial sum** *of the infinite series* (2.25).

Note: In section 2.1 we defined a sequence as a function with domain $\mathbb{N} = \{1, 2, \ldots\}$. Now, dealing with infinite series, we find it convenient to begin a sequence $\{a_k\}$ or $\{a_n\}$ with a_o rather than a_1. Thus $\{a_k\}$ stands either for $\{a_k\} = \{a_o, a_1, \ldots, a_k, \ldots\}$ or $\{a_1, a_2, \ldots, a_k, \ldots\}$, depending on the context. Similarly, we use $\{S_n\}$ to denote either $\{S_o, S_1, \ldots, S_n, \ldots\}$ or $\{S_1, S_2, \ldots, S_n, \ldots\}$.

The following theorem gives us a necessary condition for convergence of the infinite series $\sum_{k=0}^{\infty} a_k$.

Theorem 2.12 *If* $\sum_{k=0}^{\infty} a_k$ *is convergent then* $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose that $\sum_{n=0}^{\infty} a_n = S$, so that $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$. Since $S_n - S_{n-1} = a_n$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0. \quad \blacksquare$$

We can see, therefore, that

$$\lim_{n\to\infty} a_n = 0 \tag{2.26}$$

is a **necessary condition for convergence** of the infinite series $\sum_{n=0}^{\infty} a_n$.

Example 2.19
Show that the infinite series

$$\sum_{n=0}^{\infty} r^n,$$

where $|r| \geq 1$, *is divergent.*

Solution. Recall that $\lim_{n\to\infty} r^n = \infty$, when $|r| > 1$ and $\lim_{n\to\infty} r^n = 1$, when $|r| = 1$. Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n \neq 0$$

and the necessary condition for convergence is not satisfied. Consequently, $\sum r^n$ is divergent, when $|r| \geq 1$. $\blacksquare$

Example 2.20 *Find the sum of the infinite series* $\sum_{n=0}^{\infty} r^n$, $|r| < 1$.

Solution. We have

$$\begin{aligned} S_n &= 1 + r + r^2 + \cdots + r^n \\ rS_n &= \phantom{1 +{}} r + r^2 + \cdots + r^n + r^{n+1}, \end{aligned}$$

and $S_n - rS_n = 1 - r^{n+1}$ which gives

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

Since $|r| < 1$,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1 - r \lim_{n\to\infty} r^n}{1 - r} = \frac{1}{1 - r}.$$

(a) $r = .5,\ \ r = .25$ **(b)** $r = -.5,\ \ r = -.25$

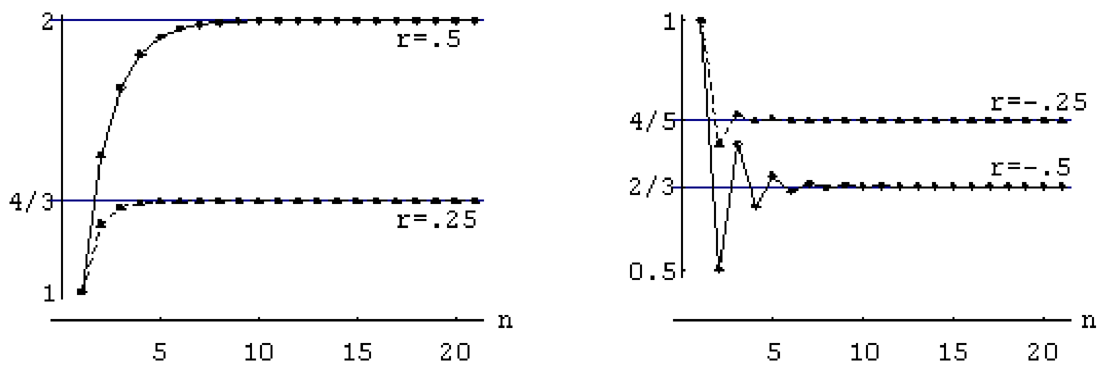


Figure 2.9: Partial sums S_n of the series $\sum_{n=0}^{\infty} r^n$ for different values of r.

Hence we have obtained the required result:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1. \tag{2.27}$$

Refer to Figure 2.9 to see the behaviour of the sequence S_n for selected values of r. ∎

2.13 Absolute Convergence of Infinite Series

Consider an infinite series $\sum_{k=0}^{\infty} |a_k|$. The sequence of its partial sums

$$S_n = |a_o| + |a_1| + |a_2| + \cdots + |a_n| \tag{2.28}$$

is clearly increasing. If $\{S_n\}$ converges, then $\{S_n\}$ must be bounded. Conversely, if $\{S_n\}$ is bounded, then, being monotone increasing, $\{S_n\}$ converges. Therefore the following theorem holds.

Theorem 2.13 *The infinite series* $\sum_{n=0}^{\infty} |a_n|$ *converges if and only if its sequence of partial sums* (2.28) *is bounded.*

Series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} |a_n|$ is convergent are very important in the theory of series.

Definition 2.11 *A series* $\sum_{n=1}^{\infty} a_n$ *such that* $\sum_{n=1}^{\infty} |a_n|$ *is convergent is called* **absolutely convergent**.

Theorem 2.14 *Absolutely convergent series are convergent.*

Proof. Let

$$S_n = \sum_{k=1}^{n} a_k, \quad T_n = \sum_{k=1}^{n} |a_k|.$$

We know that $\{T_n\}$ is a Cauchy sequence. Now, if $m > n$, we have

$$|T_n - T_m| = |a_{n+1}| + \cdots + |a_m|$$

$$|S_n - S_m| = |a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots + |a_m|.$$

Thus, for all n, m, we have

$$|S_n - S_m| \leq |T_n - T_m|.$$

Hence $\{S_n\}$ is a Cauchy sequence, so that $\lim_{n\to\infty} S_n$ exists. ∎

The theorem we give next is a simple test for convergence of infinite series, if some convergent series are available for comparison.

Theorem 2.15 The Comparison Test

Suppose that

$$0 \leq a_k \leq b_k, \quad k = 0, 1, 2, \ldots$$

Then, if $\sum_{k=0}^{\infty} b_k$ converges, so does $\sum_{k=0}^{\infty} a_k$.

Proof. Let S_n and T_n denote the n-th partial sums of $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, respectively:

$$S_n = a_o + a_1 + a_2 + \cdots + a_n,$$

$$T_n = b_o + b_1 + b_2 + \cdots + b_n.$$

Then

$$0 \leq S_n \leq T_n, \quad n = 1, 2, \ldots \tag{2.29}$$

By hypothesis, $\sum_{n=0}^{\infty} b_n$ converges, so $\{T_n\}$ is bounded and (2.29) implies that $\{S_n\}$ is also bounded. Hence, $\{S_n\}$ is nondecreasing and bounded and, by Theorem 2.13, $\{S_n\}$ converges. This completes the proof. ∎

Example 2.21 *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$ *converges.*

Solution. We have

$$\left| \frac{(-1)^{n+1}}{n(n+1)} \right| = \frac{1}{n(n+1)}.$$

Also $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, since

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so that

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1, \quad \text{as} \quad n \to \infty.$$

Thus, the given series converges. ∎

2.14 Exercises

2.1 Use the Cauchy definition of the limit of a sequence to show the following.

$$(i) \ \lim_{n\to\infty} \frac{2n-3}{3n+1} = \frac{2}{3} \qquad (ii) \ \lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} = 2 \qquad (iii) \ \lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}$$

2.2 Show that

$$(i) \ \lim_{n\to\infty} \sqrt[n]{5} = 1 \qquad (ii) \ \lim_{n\to\infty} \sqrt[n]{n^2} = 1.$$

2.3 Show that

$$(i) \quad \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$$

$$(ii) \quad \lim_{n\to\infty} \sqrt[n]{5^n + 7^n} = 7$$

$$(iii) \quad \lim_{n\to\infty} \sqrt{\alpha^n + \beta^n} = \max(\alpha, \beta), \quad \text{where} \quad \alpha \geq 0, \ \beta \geq 0.$$

2.4 If $\{a_n\} \to a$ as $n \to \infty$ then prove that $|a_n| \to |a|$ as $n \to \infty$.

2.5 Let $a_n > 0$, $n = 1, 2, \ldots$. Prove that $a_n \to 0$ if and only if $\frac{1}{a_n} \to \infty$.

2.6 **(a)** Suppose $\{a_n\}$ increases and is unbounded above. Prove that $\{a_n\} \to +\infty$ as $n \to \infty$.

(b) Suppose $\{a_n\}$ decreases and is unbounded below. Prove that $\{a_n\} \to -\infty$ as $n \to \infty$.

2.7 Let x_n be given by

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + \sqrt{x_n}}, \quad n \geq 1.$$

Show that $\{x_n\}$ converges and determine its limit.

2.8 Suppose that $0 < \alpha < 1$ and that $\{a_n\}$ is a sequence which satisfies the condition

$$|a_{n+1} - a_n| \leq \alpha^n, \quad n = 1, 2, \ldots$$

Prove that $\{a_n\}$ is a Cauchy sequence and hence converges.

2.9 Let $\alpha > 0$. Let $x_1 > \sqrt{\alpha}$ and define $\{x_n\}$ by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right), \quad n \geq 2.$$

(a) Show that $0 < x_{n+1} < x_n$ for $n \geq 2$.

(b) Deduce that $\lim_{n \to \infty} a_n$ exists and equals $\sqrt{\alpha}$.

(c) Put $\varepsilon_n = x_n - \sqrt{\alpha}$. Show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}, \qquad \text{so that} \qquad \varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}, \quad n = 1, 2, \ldots$$

(d) If $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < 1/10$, hence $\varepsilon_5 < 4 \cdot 10^{-16}$, $\varepsilon_6 < 4 \cdot 10^{-32}$.
What is the significance of these last two inequalities?
How many iterates are required to calculate $\sqrt{3}$ to 5 decimal places? Give $\sqrt{3}$ correct to 5 decimal places.

2.10 For each of the following sequences determine $\liminf a_n$ and $\limsup a_n$.

(i) $a_n = 2 - \dfrac{1}{n}$ $\qquad (ii)$ $a_n = \dfrac{1 + (-1)^n}{3}$ $\qquad (iii)$ $a_n = 2 + (-1)^n \dfrac{n}{2n+1}$

(iv) $a_n = (-1)^n + \dfrac{1}{n}$ $\qquad (v)$ $a_n = (-1)^n \dfrac{n^2}{n+1}$ $\qquad (vi)$ $a_n = 3 + \dfrac{(-1)^n}{n}$

2.11 Let $a_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$, $n = 1, 2, \ldots$. Determine

$$\limsup_{n \to \infty} a_n \quad \text{and} \quad \liminf_{n \to \infty} a_n.$$

Show that these are not the same as the numbers

$$\sup_{n \geq 1} a_n \quad \text{and} \quad \inf_{n \geq 1} a_n.$$

2.12 Show that $\{a_n\}$ converges if and only if $\liminf a_n = \limsup a_n$.

2.13 Prove the following lemma.

Lemma 2.3 *Suppose that* $\lim_{n\to\infty} x_n = \alpha > 0$ *and* $\limsup_{n\to\infty} y_n = \beta$. *Then*

$$\limsup_{n\to\infty} (x_n y_n) = \alpha\beta.$$

2.14 Show that $\lim_{n\to\infty} a_n = a$ if and only if every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ has a subsequence which converges to a.

Chapter 3

Real-Valued Functions I

3.1 Bounded Functions

Definition 3.1 *A function f defined on a set $I \subset \mathbb{R}$ is called* **bounded** *if there is a real number M such that*

$$|f(x)| \leq M \quad \text{for every} \quad x \in I.$$

Example 3.1 *Show that the function*

$$f(x) = 2x^2 + 3$$

is unbounded on $\mathbb{R}$ but is bounded on each bounded interval $I \subset \mathbb{R}$.

Solution.
Suppose that $f(x)$ is bounded on $\mathbb{R}$. Then there exists a real number $M > 0$ such that

$$|f(x)| = |2x^2 + 3| < M \quad \text{for every} \quad x \in \mathbb{R}.$$

Now, let $x = M + 1$. Then

$$|f(x)| = |f(M+1)| = |2(M+1)^2 + 3| > 2(M+1)^2 > (M+1)^2 > M + 1 > M.$$

Hence, there exists $x \in \mathbb{R}$, namely $x = M + 1$, for which $|f(x)| > M$. This contradicts our assumption that the function $f(x)$ is bounded by the number M.

If we restrict the domain of the function $f(x)$ to a bounded interval I, say $I = [a, b]$, then clearly $f(x)$ is bounded on I:

$$\forall x \in I \quad |f(x)| = |2x^2 + 3| \leq \max(2a^2 + 3, \ \ 2b^2 + 3). \ \blacksquare$$

Theorem 3.1 *If f and g are each bounded on $I \subset \mathbb{R}$ and k is a real number then the functions*

$$f + g, \quad kf, \quad \text{and} \quad f \cdot g$$

are each bounded on I.

Proof. The functions f and g are assumed to be bounded, so that there exist real numbers M_1 and M_2 such that

$$|f(x)| \leq M_1 \quad \text{and} \quad |g(x)| \leq M_2 \quad \text{for every} \quad x \in I.$$

Therefore, for every $x \in I$, we have

(i) $|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2$,

(ii) $|(kf)(x)| = |kf(x)| = |k||f(x)| \leq |k|M_1$,

(iii) $|(f \cdot g)(x)| = |f(x)g(x)| = |f(x)||g(x)| \leq M_1 \cdot M_2$. ∎

3.2 Supremum and Infimum of a Bounded Function

If f is bounded above on the set S, then by the completeness property of $\mathbb{R}$, we conclude that f has a least upper bound. This least upper bound is called the **supremum** of f on S and is denoted by

$$\sup_{x \in S} f(x), \quad \text{also} \quad \sup\{f(x) \mid x \in S\}.$$

Similarly, a function f that is bounded below has a greatest lower bound, which is called the **infimum** of f on S and is denoted by

$$\inf_{x \in S} f(x), \quad \text{also} \quad \inf\{f(x) \mid x \in S\}.$$

Thus, $M = \sup_{x \in S} f(x)$ if and only if

(i) M is an upper bound of f on S,

(ii) M is the smallest lower bound of f on S.

Similarly, $m = \inf_{x \in S} f(x)$ if and only if

(i) m is a lower bound of f on S,

(ii) m is the greatest lower bound of f on S.

Formally, we have the following definition.

Definition 3.2

$$M = \sup_{x \in S} f(x) \iff \begin{cases} \textbf{(i)} & \forall x \in S \quad f(x) \leq M, \\ \textbf{(ii)} & \forall \varepsilon > 0 \ \exists x_1 \in S \quad f(x_1) > M - \varepsilon. \end{cases}$$

$$m = \inf_{x \in S} f(x) \iff \begin{cases} \textbf{(i)} & \forall x \in S \quad f(x) \geq m, \\ \textbf{(ii)} & \forall \varepsilon > 0 \ \exists x_1 \in S \quad f(x_1) < m + \varepsilon. \end{cases}$$

Example 3.2 *Find infimum and supremum of the function* $f(x) = \dfrac{1}{x^2}$ *on the set* $S = [1, 2]$.

Solution. Evidently,

$$1 \le x \le 2 \implies \frac{1}{4} \le \frac{1}{x^2} \le 1.$$

Thus, any number $y_1 \le \frac{1}{4}$ is a lower bound of f on the set S and any number $y_2 \ge 1$ is an upper bound of f on S. Hence

$$\inf_{x\in S} f(x) = \inf_{1\le x\le 2} \left(\frac{1}{x^2}\right) = \frac{1}{4},$$

$$\sup_{x\in S} f(x) = \sup_{1\le x\le 2} \left(\frac{1}{x^2}\right) = 1. \quad \blacksquare$$

3.3 Minimum and Maximum of a Bounded Function

As we have seen above, any bounded function f on a given set S has an infimum and supremum. If $m = \inf_{x\in S} f(x)$ belongs to the range of f, that is if there exists $x_1 \in S$ such that $m = f(x_1)$, then m is the minimum value of f on S. In this case we say that f **attains its minimum** at the point $x = x_1$ and write

$$m = f(x_1) = \min_{x\in S} f(x).$$

Analogously, if there exists $x_2 \in S$ such that $M = \sup_{x\in S} f(x) = f(x_2)$, then we say that f **attains is maximum** at the point $x = x_2$ and write

$$M = f(x_2) = \max_{x\in S} f(x).$$

If $f(x_1) = \min_{x\in S} f(x)$ and $f(x_2) = \max_{x\in S} f(x)$, then, for all $x \in S$, we have

$$f(x_1) \le f(x) \le f(x_2).$$

We realize that a bounded function on S has exactly one maximum value on S, but there could be several different points at which $f(x)$ attains its maximum. A similar comment applies to infima.

Example 3.3 *Relating the concepts of* $\inf f(x)$ *and* $\sup f(x)$ *to the concepts of* $\min f(x)$ *and* $\max f(x)$. *— A bounded function does not necessarily attain a minimum or maximum.*

(a) Consider the function $f(x) = \dfrac{1}{x^2}$ on a finite closed interval $[a, b]$ that does not contain the point $x = 0$. Refer to Figure 3.1(a). Clearly,

$$\inf_{[a,b]} f(x) = \min_{[a,b]} f(x) = f(b),$$

$$\sup_{[a,b]} f(x) = \max_{[a,b]} f(x) = f(a).$$

(b) Consider the function

$$f(x) = \frac{x}{1+x^2}, \quad -\infty < x < \infty$$

and refer to Figure 3.1(b).

The function maps the domain $S = (-\infty, \infty)$ onto the interval $[-\frac{1}{2}, \frac{1}{2}]$. Thus

$$m = \inf_{x \in S} f(x) = \min_{x \in S} f(x) = f(-1) = -\frac{1}{2}$$

and

$$M = \sup_{x \in S} f(x) = \max_{x \in S} f(x) = f(1) = \frac{1}{2}.$$

Hence, f attains its minimum $m = -\frac{1}{2}$ at the point $x = -1$ and its maximum $M = \frac{1}{2}$ at the point $x = 1$.

(c) Consider the function

$$f(x) = \arctan x, \quad -\infty < x < \infty$$

and refer to Figure 3.1(c). Clearly,

$$\inf f(x) = -\frac{\pi}{2}, \quad \sup f(x) = \frac{\pi}{2},$$

but neither $-\pi/2$ nor $\pi/2$ is ever attained. ∎

(a) $f(x) = \frac{1}{x^2}$, $\inf_{x \in [a,b]} = f(b)$, $\sup_{x \in [a,b]} = f(a)$

(b) $f(x) = \frac{x}{1+x^2}$, $\inf_{x \in (-\infty,\infty)} = -\frac{1}{2}$, $\sup_{x \in (-\infty,\infty)} = \frac{1}{2}$

(c) $f(x) = \arctan x$, $\inf_{x \in (-\infty,\infty)} = -\frac{\pi}{2}$, $\sup_{x \in (-\infty,\infty)} = \frac{\pi}{2}$

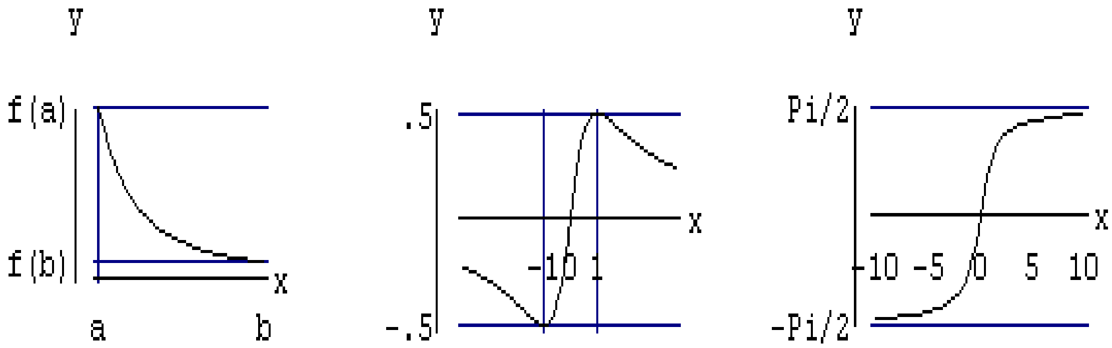


Figure 3.1: Illustrating maxima, minima, sup, and inf.

3.4 Definition of a Monotone Function

Definition 3.3 *Let f be a function defined on a set $A \subseteq \mathbb{R}$: $f : A \mapsto \mathbb{R}$.*

(a) *f is said to be* **increasing** *on A if*

$$(x_1,\ x_2 \in A \ \&\ x_1 \leq x_2) \quad \Longrightarrow \quad f(x_1) \leq f(x_2).$$

(b) *f is said to be* **strictly increasing** *on A if*

$$(x_1,\ x_2 \in A \ \&\ x_1 < x_2) \quad \Longrightarrow \quad f(x_1) < f(x_2).$$

(c) *f is said to be* **decreasing** *on A if*

$$(x_1,\ x_2 \in A \ \&\ x_1 \leq x_2) \quad \Longrightarrow \quad f(x_1) \geq f(x_2).$$

(d) *f is said to be* **strictly decreasing** *on A if*

$$(x_1,\ x_2 \in A \ \&\ x_1 < x_2) \quad \Longrightarrow \quad f(x_1) > f(x_2).$$

(e) *If f is either increasing or decreasing on A, then f is said to be* **monotone** *on A.*

(f) *If f is either strictly increasing or strictly decreasing on A, then f is said be* **strictly monotone** *on A.*

3.5 The Limit of a Function

It is important to investigate the behaviour of a function $f(x)$ for values of x close to, but not equal to, x_o, where x_o is a given point at which f is not necessarily defined. We shall require some preliminary definitions before formulating the concept of limit for functions.

Definition 3.4 .

(i) *The open interval*

$$(x_o - \delta, x_o + \delta) = \{x|\ \ x_o - \delta < x < x_o + \delta\} = \{x|\ \ |x - x_o| < \delta\}$$

is called a δ - **neighbourhood** *of the point x_o.*

(ii) *The set*

$$(x_o - \delta, x_o) \cup (x_o, x_o + \delta) = \{x|\ \ 0 < |x - x_o| < \delta\}$$

is called a **deleted** *δ -* **neighbourhood** *of the point x_o.*

Consider a function $f(x)$ defined on an interval $I \subseteq \mathbb{R}$, except possibly for some point $x_o \in I$. We say that $f(x)$ tends or converges to the limit l, as x tends to x_o ($f(x)$ has a limit l at x_o) and write

$$\lim_{x \longrightarrow x_o} f(x) = l$$

or

$$f(x) \longrightarrow l \quad \text{as} \quad x \longrightarrow x_o,$$

if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for all values of x in the δ-neighbourhood of the point x_o.

3.5.1 Cauchy Definition of Limit of a Function

Definition 3.5

$$\lim_{x\to x_o} f(x) = l \iff \forall\varepsilon > 0\ \exists\delta > 0\ \ (0 < |x - x_o| < \delta \implies |f(x) - l| < \varepsilon).$$

Intuitively, the function $f(x)$ converges to the limit l, as $x \to x_o$, if we can make $f(x)$ ***as close as we like*** to l by choosing x sufficiently close to, but not necessarily equal to, x_o.

In order to prove that $f(x)$ has limit l as $x \to x_o$, we are required to find, for any given value of $\varepsilon > 0$, a deleted δ neighbourhood of the point x_o on which $|f(x) - l| < \varepsilon$.

Example 3.4 *Applying the Cauchy definition to show that* $\lim_{x\to x_o} f(x) = l$.

(a) $\displaystyle\lim_{x\to 3} \frac{x^2 - 9}{3(x-3)} = 2.$

Let ε be given. We want to find a $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |f(x) - l| < \varepsilon, \tag{3.1}$$

where $l = 2$ and

$$f(x) = \frac{x^2 - 9}{3(x-3)}.$$

Now

$$|f(x) - l| = |\frac{x^2 - 9}{3(x-3)} - 2| = |\frac{x+3}{3} - 2| = \frac{1}{3}|x - 3|$$

and $|f(x) - l|$ will be less than ε if $|x - 3| < 3\varepsilon$. Hence, by choosing $\delta = 3\varepsilon$ we ensure that (3.1) holds. This proves that $\lim_{x\to 3} f(x) = 2$.

(b) $\displaystyle\lim_{x\to -4} (2x^2 + 3x - 4) = 16.$

Let ε be given. We want to find a $\delta > 0$ such that

$$0 < |x - (-4)| = |x + 4| < \delta \implies |f(x) - l| < \varepsilon, \tag{3.2}$$

where $l = 16$ and $f(x) = 2x^2 + 3x - 4$. We have

$$|f(x) - l| = |(2x^2 + 3x - 4) - 16| = |2x^2 + 3x - 20| = |x + 4||2x - 5|.$$

Thus

$$|x + 4| < \delta \implies |f(x) - l| < |2x - 5|\delta.$$

Now, the condition $|x+4| < \delta$ imposes some restrictions on the value of $|2x-5|$. Note that, once a suitable δ has been found, any positive number $c < \delta$ would also be appropriate. We can thus assume that $\delta \leq 1$. Then

$$\begin{aligned} |x+4| < \delta \quad &\Longrightarrow \quad |x+4| < 1 \\ &\Longrightarrow \quad -1 < x+4 < 1 \\ &\Longrightarrow \quad -5 < x < -3 \\ &\Longrightarrow \quad -15 < 2x-5 < -11 \\ &\Longrightarrow \quad -15 < 2x-5 < 15 \\ &\Longleftrightarrow \quad |2x-5| < 15, \end{aligned}$$

and we have

$$|f(x) - l| < |2x-5|\ \delta < 15\ \delta < \varepsilon \quad \text{provided that} \quad \delta \leq 1 \quad \text{and} \quad \delta \leq \frac{\varepsilon}{15}.$$

Choose

$$\delta = \min(1, \frac{\varepsilon}{15}),$$

so that

$$\delta = \begin{cases} \dfrac{\varepsilon}{15} & \text{if } \varepsilon < 15 \\ 1 & \text{if } \varepsilon \geq 15 \end{cases}$$

Then

$$|f(x) - l| < 15\delta \leq \varepsilon,$$

since $\varepsilon < 15 \quad \Longrightarrow \quad 15\delta = \varepsilon$ and $\varepsilon \geq 15 \quad \Longrightarrow \quad 15\delta = 15 \leq \varepsilon$.

Therefore, we have shown that, given any $\varepsilon > 0$, there exists δ, namely:

$$\delta = \min(1, \frac{\varepsilon}{15}),$$

such that (3.2) is satisfied.

Note that the assumption $\delta \leq 1$, we made above, is arbitrary. In fact we can assume that $\delta \leq c$, where c is any positive number. Consequently, there are many other possibilities for δ. One of these is

$$\delta = \min(\frac{1}{2}, \frac{\varepsilon}{14}),$$

since for $\delta \leq \frac{1}{2}$, we have

$$\begin{aligned} |x+4| < \frac{1}{2} \quad &\Longrightarrow \quad -\frac{1}{2} < x+4 < \frac{1}{2} \\ &\Longrightarrow \quad -\frac{9}{2} < x < -\frac{7}{2} \\ &\Longrightarrow \quad -14 < 2x-5 < -12 \\ &\Longrightarrow \quad -14 < 2x-5 < 14 \\ &\Longleftrightarrow \quad |2x-5| < 14. \end{aligned}$$

Therefore, given any $\varepsilon > 0$,

$$\delta = \min(\frac{1}{2}, \frac{\varepsilon}{14}),$$

also satisfies (3.2), since

$$|x+4| < \delta \quad \Longrightarrow \quad |f(x) - l| = |x+4||2x-5| < 14\delta \leq \varepsilon. \quad \blacksquare$$

3.5.2 Sequential (Heine) Definition of Limit of a Function

Definition 3.6 *A function $f(x)$ defined on an interval $I \subseteq \mathbb{R}$, except possibly for some point $x_o \in I$, has a limit l at x_o if and only if for every sequence $\{x_n\}$ of points of I such that $x_n \neq x_o$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} x_n = x_o$, it is true that*

$$\lim_{n\to\infty} f(x_n) = l,$$

that is,

$$\forall \{x_n\} \subset I \quad (x_n \neq x_o, \;\; n = 1, 2, \ldots, \; \& \; \lim_{n\to\infty} x_n = x_o) \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = l.$$

Theorem 3.2 *The definitions of limit of Cauchy and Heine are equivalent.*

Proof.
Assume that the function $f(x)$ satisfies the conditions of Definition 3.5, so that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \quad (0 < |x - x_o| < \delta \quad \Longrightarrow \quad |f(x) - f(x_o)| < \varepsilon)\,. \tag{3.3}$$

Let $\varepsilon > 0$ be given and consider a sequence $\{x_n\}$, $(x_n \neq x_o, \;\; n = 1, 2, \ldots)$ of points of I that converges to x_o, so that

$$\forall \varepsilon_1 > 0 \;\; \exists N_1 \in \mathbb{N} \quad (n > N_1 \quad \Longrightarrow \quad 0 < |x_n - x_o| < \varepsilon_1).$$

Since the above statement holds for every $\varepsilon_1 > 0$, it does so for $\varepsilon_1 = \delta$. Hence there is N_1 such that

$$0 < |x_n - x_o| < \delta \quad \text{for} \quad n > N_1$$

and, by (3.3), we conclude that

$$|f(x_n) - l| < \varepsilon \quad \text{for} \quad n > N_1.$$

Therefore, for every $\varepsilon > 0$, there exists $N = N_1$, such that

$$n > N \quad \Longrightarrow \quad |f(x_n) - l| < \varepsilon,$$

which means that the sequence $\{f(x_n)\}$ converges to the limit l.

Conversely, assume that the function $f(x)$ satisfies the conditions of Definition 3.6, so that for each sequence $\{x_n\}$ $(x_n \neq x_o,\; n = 1, 2, \ldots)$ of points of I that converges to x_o, as $n \to \infty$, the corresponding sequence $\{f(x_n)\}$ converges to l , as $n \to \infty$:

$$\forall \{x_n\} \in I \quad (x_n \neq x_o, \;\; n = 1, 2, \ldots \; \& \; \lim_{n\to\infty} x_n = x_o) \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = l. \tag{3.4}$$

Now we suppose that it is not true that (3.3) holds. Hence, there is $\varepsilon = \varepsilon_o$ such that for each value of $\delta > 0$ we can find $x,\ x \neq x_o$, satisfying

$$0 < |x - x_o| < \delta \quad \text{and} \quad |f(x) - l| \geq \varepsilon_o.$$

In particular, let $\delta = \dfrac{1}{n}, n \in \mathbb{N}$. Then we can find an $x_n,\ (n = 1, 2, \ldots)$ satisfying both inequalities:

$$0 < |x_n - x_o| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - l| \geq \varepsilon_o.$$

The sequence $\{x_n\}$ so constructed converges to x_o. Therefore, there exists a sequence, $\{x_n\}$, of points of I such that $x_n \neq x_o,\ n = 1, 2, \ldots,$ with $\lim\limits_{n\to\infty} x_n = x_o$, and there exists an $\varepsilon = \varepsilon_o > 0$ such that $|f(x_n) - l| \geq \varepsilon$ for all $n \in \mathbb{N}$ which implies that $\lim\limits_{n\to\infty} f(x_n) \neq l$. This contradicts our assumption (3.4). ∎

Example 3.5 *Prove that* $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} f(x^3)$.

Solution.
Let $\lim\limits_{x\to 0} f(x) = L$. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in I \quad (0 < |x| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon)\,.$$

Now, if $0 < |x| < \min(1,\ \delta)$ then

$$\left(0 < |x^3| < \delta\right) \quad \Longrightarrow \quad |f(x^3) - L| < \varepsilon,$$

and we conclude that $\lim\limits_{x\to 0} f(x^3) = L$.

On the other hand, assume that $\lim\limits_{x\to 0} f(x^3) = K$. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in I \quad \left(0 < |x| < \delta \quad \Longrightarrow \quad |f(x^3) - K| < \varepsilon\right).$$

If $0 < |x| < \delta^3$ then $0 < |\sqrt[3]{x}| < \delta$, so

$$|f[(\sqrt[3]{x})^3] - K| < \varepsilon, \quad \text{hence} \quad |f(x) - K| < \varepsilon.$$

Hence, $\lim\limits_{x\to 0} f(x) = K$. ∎

Example 3.6 *An example of a function where* $\lim\limits_{x\to 0} f(x^2)$ *exists, but* $\lim\limits_{x\to 0} f(x)$ *does not.*

Solution.
Let

$$f(x) = \begin{cases} -2 & \text{for } x < 0 \\ \ \ 2 & \text{for } x \geq 0 \end{cases}$$

Then $f(x^2) = 2$, for $-\infty < x < \infty$ and $\lim\limits_{x\to 0} f(x^2) = 2$ but $\lim\limits_{x\to 0} f(x)$ does not exist. ∎

3.6 Limits from the Left and Limits from the Right

Suppose that $f(x)$ is defined on an interval $(a,\ x_o)$ or $(x_o,\ b)$, but not necessarily defined in a neighbourhood of the point x_o. Then the concept of limit of $f(x)$, as $x \to x_o$, defined earlier is not applicable. We can consider, however, one-sided limits as defined below. If, for example, f is defined on an open interval (a, b), then we would be interested in the behaviour of $f(x)$, as x approaches b from the left or as x approaches a from the right.

Definition 3.7 .

(i) *We say that* $\lim\limits_{x\to a+} f(x) = A$ *if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \quad (a < x < a + \delta \quad \Longrightarrow \quad |f(x) - A| < \varepsilon).$$

(ii) *We say that* $\lim\limits_{x\to b-} f(x) = B$ *if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \quad (b - \delta < x < b \quad \Longrightarrow \quad |f(x) - B| < \varepsilon).$$

Theorem 3.3 *Let $f(x)$ be defined on an interval I except possibly at $a \in I$. $\lim\limits_{x\to a} f(x)$ exists if and only if*

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x).$$

Proof.

Suppose that $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x) = K$. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \ \forall x \in I \quad (a - \delta < x < a + \delta \quad \Longrightarrow \quad |f(x) - K| < \varepsilon).$$

This implies that $\forall \varepsilon > 0\ \ \exists \delta > 0\ \ \ \forall x \in I$

$$a < x < a + \delta \quad \Longrightarrow \quad |f(x) - K| < \varepsilon \quad \text{and} \quad a - \delta < x < a \quad \Longrightarrow \quad |f(x) - K| < \varepsilon,$$

which means that $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to a-} f(x)$ exist and $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a-} f(x) = K$.

On the other hand, assume that

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L.$$

Then

$$\forall \varepsilon_1 > 0\ \exists \delta_1 > 0\ \ \forall x \in I \quad (a < x < a + \delta_1 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon_1)$$

and

$$\forall \varepsilon_2 > 0\ \exists \delta_2 > 0\ \ \forall x \in I \quad (a - \delta_2 < x < a \quad \Longrightarrow \quad |f(x) - L| < \varepsilon_2).$$

Let $\varepsilon = \varepsilon_1 = \varepsilon_2 > 0$ be given and let $\delta = \min(\delta_1,\ \delta_2)$. If $0 < |x - a| < \delta$, then either $a - \delta_2 \le a - \delta < x < a$ or else $a < x < a + \delta \le a + \delta_1$. In both cases, we have $|f(x) - L| < \varepsilon$. Thus $\lim\limits_{x\to a} f(x) = L$. ∎

Theorem 3.4 *Let f be a function defined on the interval $I = [a, b] \subset \mathbb{R}$ and let c be an interior point of I, $a < c < b$.*

(a) *If f is increasing on I, then*

(i) $\lim\limits_{x \to c-} f(x) = \sup\limits_{a \le x < c} f(x)$,

(ii) $\lim\limits_{x \to c+} f(x) = \inf\limits_{c < x \le b} f(x)$.

(b) *If f is decreasing on I, then*

(i) $\lim\limits_{x \to c-} f(x) = \inf\limits_{a \le x < c} f(x)$,

(ii) $\lim\limits_{x \to c+} f(x) = \sup\limits_{c < x \le b} f(x)$.

Proof. We shall give a detailed proof for the statement **(i)** of case **(a)**. The proofs of the remaining statements are similar.

Since f is increasing on I, we have

$$(x \in I \ \& \ x < c) \quad \Longrightarrow \quad f(x) \le f(c).$$

Let L be the supremum of the set

$$S = \{f(x) \mid x \in I, \ x < c\},$$

which is clearly bounded above by $f(c)$.

Given $\varepsilon > 0$, there exists $y_\varepsilon \in I$, $y_\varepsilon < c$, such that $L - \varepsilon < f(y_\varepsilon) \le L$, since L is the smallest upper bound of S.

Let $\delta = c - y_\varepsilon$ and consider any y in the interval (y_ε, c). Since f is increasing, we have

$$0 < c - y < \delta \quad \Longrightarrow \quad y_\varepsilon < y < c \quad \Longrightarrow \quad L - \varepsilon < f(y_\varepsilon) \le f(y) \le L.$$

Therefore

$$0 < c - y < \delta \quad \Longrightarrow \quad |f(y) - L| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim\limits_{x \to c-} f(x) = L$, as required. ∎

3.7 Properties of Limits of Functions

Theorem 3.5 *Let $f(x)$ and $g(x)$ be defined on an interval I except possibly at $a \in I$. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. Then:*

(i) $\lim_{x\to a}(\alpha f(x)) = \alpha L$;

(ii) $\lim_{x\to a}(f(x) + g(x)) = L + K$;

(iii) $\lim_{x\to a} f(x)g(x) = LK$;

(iv) $\lim_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{K}$, *provided that $K \neq 0$;*

(v) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, *provided that $K \neq 0$.*

Proof. By assumption, $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, so that

$$\forall \varepsilon_1 > 0 \;\; \exists \delta_1 > 0 \;\; \forall x \quad (0 < |x-a| < \delta_1 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon_1), \tag{3.5}$$

and

$$\forall \varepsilon_2 > 0 \;\; \exists \delta_2 > 0 \;\; \forall x \quad (0 < |x-a| < \delta_2 \quad \Longrightarrow \quad |g(x) - K| < \varepsilon_2). \tag{3.6}$$

(i) We have

$$|\alpha f(x) - \alpha L| = |\alpha||f(x) - L|, \quad x \in I.$$

Assume $\alpha \neq 0$. Let $\varepsilon > 0$ be chosen. By (3.5), given any $\varepsilon_1 = \frac{\varepsilon}{|\alpha|}$, there exists $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon_1.$$

Thus

$$0 < |x-a| < \delta_1 \quad \Longrightarrow \quad |\alpha f(x) - \alpha L| = |\alpha||f(x) - L| < |\alpha|\varepsilon_1 = |\alpha|\frac{\varepsilon}{|\alpha|} = \varepsilon.$$

This completes the proof of **(i)**, when $\alpha \neq 0$. When $\alpha = 0$, there is nothing to prove.

(ii) Let $\varepsilon > 0$ be given. Let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ and let $\delta = \min(\delta_1, \delta_2)$. Then, for x satisfying the inequality $0 < |x-a| < \delta$, both (3.5) and (3.6) hold and we get

$$|f(x)+g(x)-(L+K)| = |(f(x)-L)+(g(x)-K)| \leq |f(x)-L|+|g(x)-K| < \varepsilon_1+\varepsilon_2 = \varepsilon.$$

Hence, we have proved that

$$\forall \varepsilon > 0 \;\; \exists \delta = \min(\delta_1, \delta_2) > 0 \;\; \forall x \quad (0 < |x-a| < \delta \quad \Longrightarrow \quad |f(x)+g(x)-(L+K)| < \varepsilon),$$

which means that $\lim_{x\to a}(f(x) + g(x)) = L + K$.

(iii) We have

$$\begin{aligned} |f(x)g(x) - LK| &= |f(x)g(x) - Lg(x) + Lg(x) - LK| \\ &\le |f(x)g(x) - Lg(x)| + |Lg(x) - LK| \\ &= |g(x)||f(x) - L| + |L||g(x) - K|. \end{aligned}$$

Now, let $\varepsilon_2 = 1$ in (3.6). Then there exists $\delta_2 = \delta_{20}$ such that

$$\begin{aligned} 0 < |x-a| < \delta_{20} \quad &\Longrightarrow \quad |g(x) - K| < 1 \\ &\Longrightarrow \quad |g(x)| = |g(x) - K + K| \le |g(x) - K| + |K| < 1 + |K|. \end{aligned}$$

Let $\varepsilon > 0$ be given and let

$$\varepsilon_1 = \frac{\varepsilon}{2(1+|K|)} \quad \text{and} \quad \varepsilon_2 = \frac{\varepsilon}{2(1+|L|)}.$$

Then there exist $\delta_1 = \delta_{11} > 0$ and $\delta_2 = \delta_{21} > 0$ such that

$$\begin{aligned} |f(x)g(x) - LK| &\le |g(x)||f(x) - L| + |L||g(x) - K| \\ &< (1+|K|)\varepsilon_1 + |L|\varepsilon_2 \\ &= (1+|K|)\frac{\varepsilon}{2(1+|K|)} + L\frac{\varepsilon}{2(1+|L|)} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{L}{|L|+1} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all x satisfying the inequality

$$0 < |x-a| < \delta, \quad \text{where} \quad \delta = \min(\delta_{20}, \delta_{11}, \delta_{21}).$$

This completes the proof of **(iii)**.

(iv) We have

$$\left|\frac{1}{g(x)} - \frac{1}{K}\right| = \frac{|g(x) - K|}{|K||g(x)|}.$$

Let $\varepsilon_2 = \frac{1}{2}|K|$. By (3.6), there exists $\delta_2 = \delta_{20}$ such that

$$0 < |x-a| < \delta_{20} \quad \Longrightarrow \quad |g(x)| = |K - (K - g(x))| \ge |K| - |K - g(x)| > |K| - \frac{|K|}{2} = \frac{|K|}{2}.$$

Let $\varepsilon > 0$ be given and let $\varepsilon_2 = \frac{\varepsilon}{2}|K|^2$. By (3.6), there exists $\delta_2 = \delta_{21} > 0$ such that

$$0 < |x-a| < \delta_{21} \quad \Longrightarrow \quad |g(x) - K| < \varepsilon_2.$$

Therefore, choosing $\delta = \min(\delta_{20}, \delta_{21})$, we have

$$0 < |x-a| < \delta \quad \Longrightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{K}\right| = \frac{|g(x) - K|}{|K||g(x)|} \le \frac{2}{|K|^2}\frac{\varepsilon}{2}|K|^2 = \varepsilon.$$

This means that $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{K}$.

(v) Using **(iii)**, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \lim_{x\to a} \frac{1}{g(x)} = \frac{L}{K}. \quad \blacksquare$$

Theorem 3.6 Squeeze Theorem for the limit of a function
Let the functions $f(x)$, $g(x)$, and $h(x)$ be defined on an interval I except possibly at $a \in I$. Suppose that the following inequality holds for all $x \neq a$, $x \in I$.

$$h(x) \leq f(x) \leq g(x). \tag{3.7}$$

If $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = L$ *then* $\lim\limits_{x\to a} f(x) = L$.

Proof. By assumption, $\lim\limits_{x\to a} h(x) = L$ and $\lim\limits_{x\to a} g(x) = L$, so that

$$\forall \varepsilon_1 > 0 \;\; \exists \delta_1 > 0 \;\; \forall x \;\; (0 < |x - a| < \delta_1 \implies |h(x) - L| < \varepsilon_1), \tag{3.8}$$

and

$$\forall \varepsilon_2 > 0 \;\; \exists \delta_2 > 0 \;\; \forall x \;\; (0 < |x - a| < \delta_2 \implies |g(x) - L| < \varepsilon_2). \tag{3.9}$$

Let $\varepsilon > 0$ be given. We are to show that there exists a $\delta > 0$ such that for all $x \in I$,

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon$ in (3.8) and (3.9) above and let δ_{10} and δ_{20} be the corresponding values of δ_1 and δ_2, respectively. Then, using (3.7), (3.8), and (3.9), we conclude that

$$\exists \delta = \min(\delta_{10}, \delta_{20}) \;\; \forall x \;\; (0 < |x - a| < \delta \implies L - \varepsilon < h(x) \leq f(x) \leq g(x) < L + \varepsilon).$$

Hence $L - \varepsilon < f(x) < L + \varepsilon$ or, equivalently, $|f(x) - L| < \varepsilon$ provided that $0 < |x - a| < \delta$. Therefore $\lim\limits_{x\to a} f(x) = L$. $\blacksquare$

Example 3.7 *Show that* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Solution. Let $0 < x < \pi/2$. From the diagram, we have

$OB = OP = 1$,
arc $BP = x$,
$AP = \sin x$,
$BC = \tan x$.

Clearly, $AP <$ arc $BP < BC$ and $\sin x < x < \tan x$.
Dividing each side of the above inequality by $\sin x$ gives

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

which implies that, for $0 < x < \pi/2$,

$$\cos x < \frac{\sin x}{x} < 1.$$

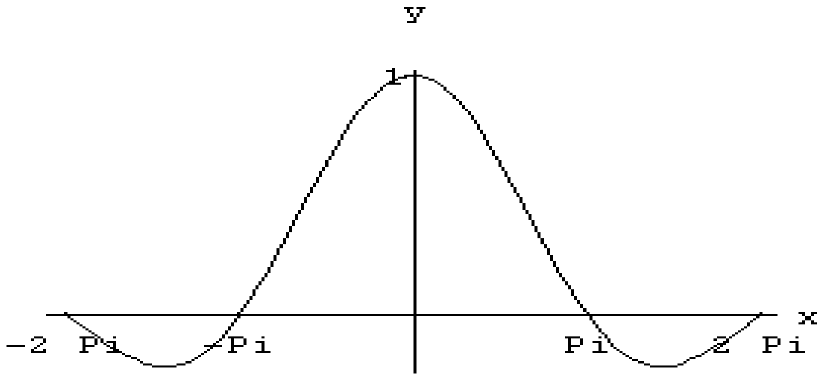


Figure 3.2: The function $f(x) = \dfrac{\sin x}{x}$ considered in Example 3.7.

Since $\cos x$ is a continuous function, $\lim_{x\to 0} \cos x = \cos 0 = 1$ and, by the Squeeze Theorem, we obtain

$$\lim_{x\to 0+} \frac{\sin x}{x} = 1.$$

To complete the solution, we note that the left-hand limit is also 1:

$$\lim_{x\to 0-} \frac{\sin x}{x} = \lim_{x\to 0+} \frac{\sin(-x)}{-x} = \lim_{x\to 0+} \frac{\sin x}{x} = 1. \quad \blacksquare$$

3.8 Continuity at a Point

In section 3.5 when we discussed the limit of a function $f(x)$ at a given point x_o, we were only interested in the behaviour of $f(x)$ as x tends to x_o, not in what happens when x equals x_o. If $\lim_{x\to x_o} f(x)$ exists, it is quite possible that $f(x)$ is not defined at the point x_o or, it is quite possible that $\lim_{x\to x_o} f(x) \neq f(x_o)$. Refer to Figure 3.3 which illustrates the following situations:

(a) $f(x)$ does not have a limit as $x \longrightarrow x_o$;

(b) $\lim_{x\to x_o} f(x) = -\infty$;

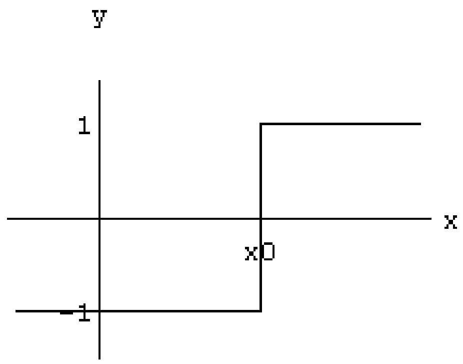
(c) $\lim_{x\to x_o} f(x) = +\infty$;

(d) $\lim_{x\to x_o} f(x)$ exists and is finite but $f(x)$ is not defined at the point x_o;

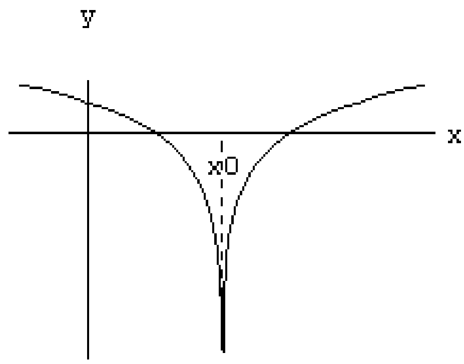
(e) $\lim_{x\to x_o} f(x) = L \neq f(x_o)$;

(f) $\lim_{x\to x_o} f(x) = L = f(x_o)$.

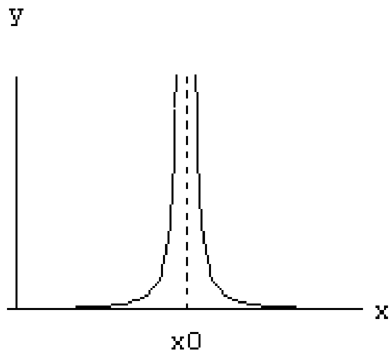
(a) $f(x) = \dfrac{|x - x_o|}{x - x_o}, \quad x \neq x_o$



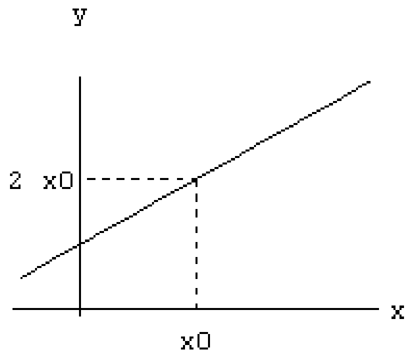
(b) $f(x) = \log|x - x_o|,$
$x \neq x_o$



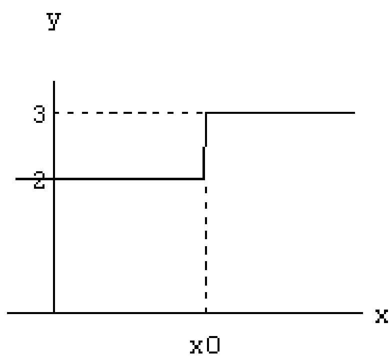
(c) $f(x) = \dfrac{1}{(x - x_o)^2},$
$x \neq x_o$



(d) $f(x) = \dfrac{x^2 - x_o^2}{x - x_o}, \quad x \neq 0$
$\lim_{x \to x_o} f(x) = 2x_o$



(e) $f(x) = \begin{cases} 3, & x > x_o \\ 2, & x \leq x_o \end{cases}$



(f) $f(x) = \begin{cases} \dfrac{\sin(x - x_o)}{x - x_o}, & x \neq x_o \\ 1, & x = x_o \end{cases}$
$\lim_{x \to x_o} f(x) = 1.$

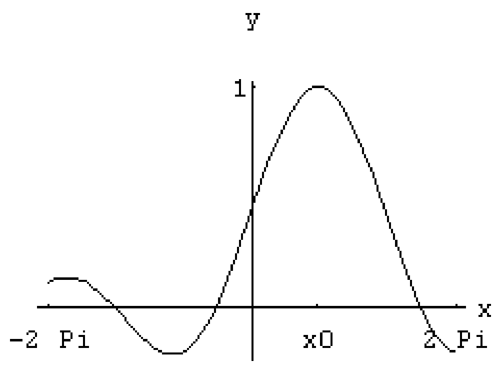


Figure 3.3: Non-continuous functions (a) — (e) versus a continuous function (f).

Definition 3.8 *The function $f(x)$ defined on an interval I is* **continuous** *at x_o, if $\lim_{x \to x_o} f(x) = f(x_o)$.*

Intuitively, a function f is continuous at the point $x = x_o$, if the graph of f does not have a "break" or a "jump" at the point x_o.

We observe that the function **(f)** in Figure 3.3 is continuous at the specified point x_o, but none of the functions **(a)** – **(e)** is continuous at x_o. Using the Cauchy and Heine definitions of the limit of a function, definition 3.8 can be written as follows.

Definition 3.9 *Let $f(x)$ be defined on an interval I and let $x_o \in I$.*

(i) *(Cauchy definition of continuity at a point.)*

$f(x)$ is **continuous** *at the point $x = x_o$, if*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \ \forall x \in I \ \ (|x - x_o| < \delta \quad \Longrightarrow \quad |f(x) - f(x_o)| < \varepsilon).$$

(ii) *(Heine definition of continuity at a point.)*

$f(x)$ is **continuous** *at the point $x = x_o$, if*

$$\forall \{x_n\} \in I \ \left(\lim_{n \to \infty} x_n = x_o \quad \Longrightarrow \quad \lim_{n \to \infty} f(x_n) = f(x_o) \right).$$

Example 3.8 *Use the Heine definition of continuity to show that $f(x) = x^2$ is continuous at x_o.*

Solution. Let $\{x_n\}$ be a sequence converging to x_o. Let $\varepsilon > 0$ be given. We require N such that

$$n > N \quad \Longrightarrow \quad |x_n^2 - x_o^2| < \varepsilon.$$

Let N_1 be such that $n > N_1 \quad \Longrightarrow \quad |x_n - x_o| < 1$. Then $|x_n| < |x_o| + 1$. Hence $|x_n + x_o| < 2|x_o| + 1$.

Let N_2 be such that $n > N_2 \quad \Longrightarrow \quad |x_n - x_o| < \dfrac{\varepsilon}{2|x_o| + 1}$.

For $n > \max(N_1, N_2)$, we have

$$|x_n^2 - x_o^2| = |x_n - x_o| \cdot |x_n + x_o| < \frac{\varepsilon}{2|x_o| + 1} \cdot (2|x_o| + 1) = \varepsilon. \ \blacksquare$$

Refer to Figure 3.3(e). The function $f(x)$ shown there is not continuous, since it does not have a limit when $x \longrightarrow x_o$. The function is defined at the point x_o and the one-sided limits exist:

$$\lim_{x \to x_o-} f(x) = 2 \quad \text{and} \quad \lim_{x \to x_o+} f(x) = 3.$$

Since $\lim_{x \to x_o-} f(x) = 2 = f(x_o)$, we say that $f(x)$ is continuous on the left of the point $x = x_o$. It is not continuous on the right of x_o since $\lim_{x \to x_o+} f(x) = 3 \neq f(x_o) = 2$.

The definition of one-sided limits leads to the definition of one-sided continuity.

Definition 3.10 *Suppose that the function $f(x)$ is defined on an interval I and let $x_o \in I$.*

(i) *If $\lim_{x\to x_o-} f(x) = f(x_o)$, we say that f is* **continuous on the left** *at the point $x = x_o$. Thus, $f(x)$ is continuous on the left at $x = x_o$, if*

$$\forall \varepsilon > 0 \;\exists \delta > 0 \;\; \forall x \in I \;\; (x_o - \delta < x \le x_o \implies |f(x) - f(x_o)| < \varepsilon)\,,$$

or

$$\forall \{x_n\} \;\; \Big(x_n \le x_o \;\&\; \lim_{n\to\infty} x_n = x_o \implies \lim_{n\to\infty} f(x_n) = f(x_o) \Big)\,.$$

(ii) *If $\lim_{x\to x_o+} f(x) = f(x_o)$, we say that f is* **continuous on the right** *at the point $x = x_o$. Thus, $f(x)$ is continuous on the right at $x = x_o$, if*

$$\forall \varepsilon > 0 \;\exists \delta > 0 \;\; \forall x \in I \;\; (x_o \le x < x_o + \delta \implies |f(x) - f(x_o)| < \varepsilon)\,,$$

or

$$\forall \{x_n\} \;\; \Big(x_n \ge x_o \;\&\; \lim_{n\to\infty} x_n = x_o \implies \lim_{n\to\infty} x_n = f(x_o) \Big)\,.$$

Note that the function $f(x)$ in Figure 3.3 (e) is left continuous.

Example 3.9 *Examining one-sided continuity.*

Refer to Figure 3.4 which shows two functions $f(x)$ and $g(x)$ that are not continuous at the point $x = 0$. To examine one-sided continuity of the functions, we evaluate the one-sided limits. We have

$$f(x) = \begin{cases} a^{1/x}, & x \ne 0 \\ 0, & x = 0, \end{cases}$$
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} a^{1/x} = \lim_{y\to +\infty} a^y = +\infty,$$
$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} a^{1/x} = \lim_{y\to -\infty} a^y = 0.$$

Thus

$$\lim_{x\to 0-} f(x) = 0 = f(0)$$

and we conclude that $f(x)$ is left-continuous at the point $x = 0$. Clearly, $f(x)$ is not right-continuous.

Now, we have

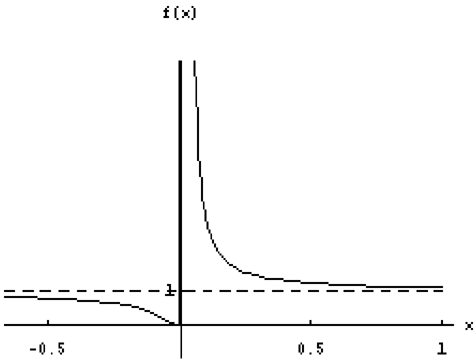
$$g(x) = \begin{cases} \arctan(1/x), & x \ne 0 \\ \pi/2, & x = 0, \end{cases}$$
$$\lim_{x\to 0+} g(x) = \lim_{x\to 0+} \arctan(1/x) = \lim_{y\to +\infty} \arctan y = \frac{\pi}{2},$$
$$\lim_{x\to 0-} g(x) = \lim_{x\to 0-} \arctan(1/x) = \lim_{y\to -\infty} \arctan y = -\frac{\pi}{2}.$$

Thus

$$\lim_{x \to 0+} g(x) = \frac{\pi}{2} = g(0)$$

and we conclude that $f(x)$ is right-continuous at the point $x = 0$. Clearly, $f(x)$ is not left-continuous.

(a) $f(x) = \begin{cases} a^{1/x}, & x \neq 0 \\ 0, & x = 0, \quad (a > 0) \end{cases}$ **(b)** $g(x) = \begin{cases} \arctan(1/x), & x \neq 0 \\ \pi/2, & x = 0, \end{cases}$



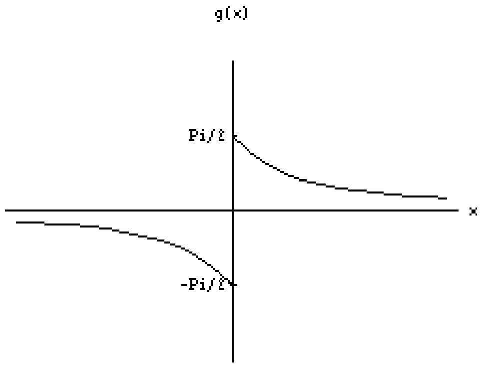


Figure 3.4: Illustrating one-sided continuity of a function.

Theorem 3.7 *A function f is continuous at $x = x_o$ if and only if f is both left-continuous and right-continuous at x_o.*

3.9 Algebra of Continuous Functions

Theorem 3.8 *Assume that the functions $f(x)$ and $g(x)$ defined on an interval I are continuous at a given point $a \in I$. Then the following rules hold.*

(i) Sum Rule: *The function $f + g$ is continuous at $x = a$.*

(ii) Product Rule: *The function $f \cdot g$ is continuous at $x = a$.*

(iii) Quotient rule: *If $g(a) \neq 0$ then the function $\dfrac{f}{g}$ is continuous at $x = a$.*

The proofs of the above rules follow immediately from the definition of continuity at a point and the corresponding rules for limits of functions (see theorem 3.5).

Theorem 3.9 Squeeze Rule for continuous functions.

Let the functions $f(x)$, $g(x)$, and $h(x)$ be defined on an interval I and let $a \in I$. Assume that

$$h(x) \leq f(x) \leq g(x)$$

for all x in some neighbourhood of the point $x = a$ and that

$$h(a) = f(a) = g(a).$$

If $h(x)$ and $g(x)$ are continuous at $x = a$ then the function $f(x)$ is continuous at the point a.

Theorem 3.10 Composite Rule
Let f and g be continuous at a and $f(a)$ respectively, such that the composite function $g \circ f$ is defined. Then $g \circ f$ is continuous at $x = a$.

Proof. Let $f(a) = d$. Since g is continuous at d,

$$\forall \varepsilon > 0 \ \exists \delta_1 > 0 \ \ \forall y \ \ (|y - d| < \delta_1 \implies |g(y) - g(d)| < \varepsilon). \tag{3.10}$$

Since f is continuous at a,

$$\forall \varepsilon_2 > 0 \ \exists \delta_2 > 0 \ \ \forall x \ \ (|x - a| < \delta_2 \implies |f(x) - f(a)| < \varepsilon_2). \tag{3.11}$$

Since $\delta_1 > 0$, we can use $\varepsilon_2 = \delta_1$ in (3.11) to get

$$|x - a| < \delta_2 \implies |f(x) - f(a)| < \delta_1.$$

Now, $y = f(x)$, $d = f(a)$, so that (3.10) gives

$$|x - a| < \delta_2 \implies |f(x) - f(a)| < \delta_1 \implies |g(f(x)) - g(f(a))| < \varepsilon.$$

Hence

$$\forall \varepsilon > 0 \ \exists \delta = \delta_2 > 0 \ \ \forall x \ \ (|x - a| < \delta_2 \implies |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon)$$

which means that the function $g \circ f$ is continuous at the point $x = a$. ∎

3.10 Continuity on a Set

Definition 3.11 *A function f defined on an open interval $I = (a, b)$ is said to be* **continuous on** *(a, b) if it is continuous at each point $x = x_o \in (a, b)$.*

Extending the above definition to the case when I is a closed interval, $I = [a, b]$, it is natural to require f be right-continuous at $x = a$ and left-continuous at $x = b$, in addition to the requirements of that definition.

Definition 3.12 *A function f defined on a given closed interval $I = [a, b]$ is said to be* **continuous on the interval** *$[a, b]$ if*

(i) *f is continuous at each point $x = x_o \in (a, b)$;*

(ii) *f is right-continuous at the point $x = a$;*

(iii) *f is left-continuous at the point $x = b$.*

Continuity may, in fact, be defined on arbitrary subsets of $\mathbb{R}$ but we shall not need the concept in this course:

f is continuous on a set A if it is continuous at all points a of A:

$$\forall a \in A \ \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x \in A \ \ (|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon).$$

3.11 Types of Discontinuity

Example 3.10 *Consider the function* $f(x) = x \sin\left(\frac{1}{x}\right), \quad x \neq 0.$

(i) *Prove that* $f(x)$ *is continuous at any point* $x \neq 0$.

(ii) *Is it possible to find a constant c such that the function*

$$g(x) = \begin{cases} c & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

is continuous at every point $x, \quad -\infty < x < \infty$ *?*

Solution.
(i) Let

$$f_1 = x, \quad f_2 = 1, \quad f_3 = \frac{f_2(x)}{f_1(x)} = \frac{1}{x}, \; x \neq 0, \quad f_4(x) = \sin(f_3(x)) = \sin(\frac{1}{x}), \; x \neq 0.$$

Then, using theorems 3.8 and 3.10, we conclude that the functions $f_3(x)$ and $f_4(x)$ are continuous at every point $x \neq 0$. Therefore the function $f(x)$ is continuous for $x \neq 0$.

(ii) By the Squeeze Theorem, $\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$. If $c = 0$, therefore, the function

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin(\frac{1}{x}) & x \neq 0 \end{cases}$$

is continuous at every point $x, \quad -\infty < x < \infty.$ ∎

A function that is not continuous at a given point x_o is said to have a discontinuity at that point. If $f(x)$ is not defined at x_o, but the limit of $f(x)$ at x_o exists, we say that the function $f(x)$ has a **removable discontinuity** at the point x_o. Defining $f(x_o) = \lim_{x \to x_o} f(x)$ "removes" the discontinuity at x_o, as was the case in Example 3.10.

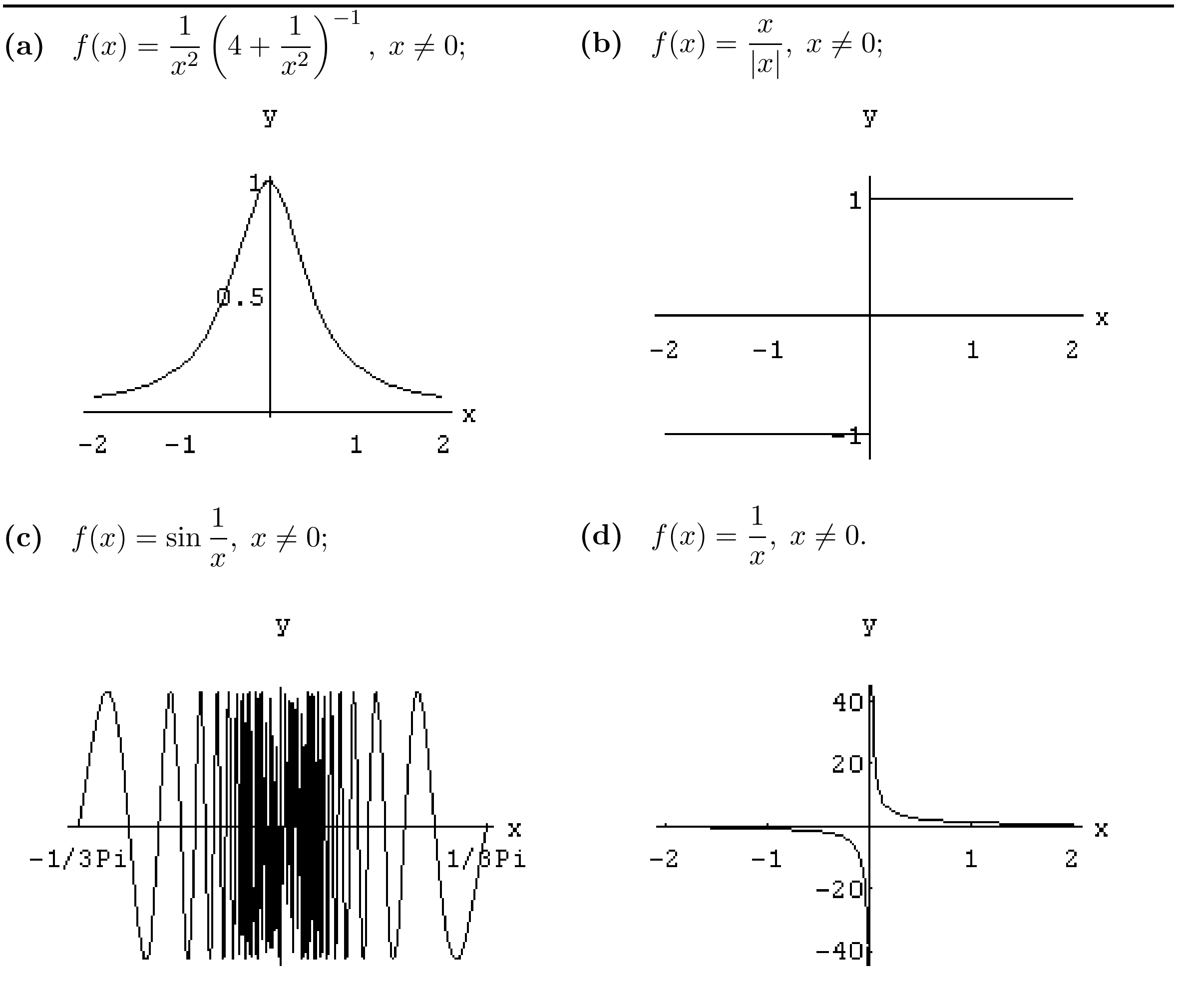


Figure 3.5: Illustrating different kinds of discontinuities of a function.

Example 3.11 *Different kinds of discontinuities of functions.*

Refer to Figure 3.4 and observe different kinds of discontinuities.

(a) removable discontinuity.

$$f(x) = \frac{1}{x^2}(4 + \frac{1}{x^2})^{-1}, \ \ x \neq 0.$$

We can see that $f(x)$ is not continuous at the point $x = 0$ for $f(x)$ is not defined at $x = 0$. $f(x)$ has a limit at $x = 0$, namely $\lim_{x \to 0} f(x) = 1$, so that we can "extend" the definition of $f(x)$ to make the new function $f_1(x)$ continuous at $x = 0$:

$$f_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2}(4 + \frac{1}{x^2})^{-1} & x \neq 0. \end{cases}$$

(b) jump discontinuity.

$$f(x) = \frac{x}{|x|}, \quad x \neq 0.$$

We note that $f(x)$ does not have a limit at $x = 0$ for

$$\lim_{x \to 0+} f(x) = +1 \quad \text{and} \quad \lim_{x \to 0-} f(x) = -1.$$

(c) oscillating discontinuity.

$$f(x) = \sin(\frac{1}{x}), \quad x \neq 0.$$

The oscillations are bounded.

(d) infinite discontinuity.

$$f(x) = \frac{1}{x}, \quad x \neq 0.$$

3.12 Exercises

3.1 Using the definition of supremum/infimum prove the following:

$$(a) \quad \sup_{-\infty < x < \infty} (x - [x]) = 1 \qquad (b) \quad \inf_{-\infty < x < \infty} (x - [x]) = 0$$

$$(c) \quad \sup_{-\infty < x < \infty} \frac{x}{4 + x^2} = \frac{1}{4} \qquad (d) \quad \inf_{-\infty < x < \infty} \frac{x}{4 + x^2} = 0.$$

3.2 Find $\sup f(x)$ and $\inf f(x)$, where they exist, for each of the following functions on the indicated domain:

(*a*) $f(x) = 3 + 2x - x^2, \;\; 0 < x < 4$ (*b*) $f(x) = \frac{3}{x}, \;\; -2 < x < -1$

(*c*) $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{3}{x} & \text{for } 0 < |x| \leq 1 \end{cases}$ (*d*) $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x \sin \dfrac{1}{x} & \text{for } 0 < x < 2\pi \end{cases}$

(*e*) $f(x) = 2 - |x - 1|, \;\; -2 < x < 2$ (*f*) $f(x) = 2x - [x], \;\; 0 < x < 10$

(*g*) $f(x) = 2 + e^{-x^2}, \;\; -\infty < x < \infty$ (*h*) $f(x) = 1 + e^{-|x|}, \;\; -\infty < x < \infty$

(*i*) $f(x) = \dfrac{x}{x - 2}, \;\; x \in (-\infty, 2) \cup (2, \infty)$ (*j*) $f(x) = \dfrac{1 - x^2}{1 + x^2}, \;\; -\infty < x < \infty$

(*k*) $f(x) = e^{-x}, \;\; -\infty < x < \infty$ (*l*) $f(x) = \exp(-\dfrac{1}{x}), \;\; x \in (-\infty, 2) \cup (2, \infty)$.

3.3 Find the extreme values for each function of the previous exercise.

3.4 Prove that if f and g are both bounded on an interval $a \leq x \leq b$, then

$$\inf f + \inf g \leq \inf(f + g) \leq \sup(f + g) \leq \sup f + \sup g.$$

3.5 Find functions f and g neither of which is bounded but such that the product $f \cdot g$ is bounded on I.

3.6 Suppose that f is bounded on I and g is unbounded on I. Prove that the sum $f+g$ in unbounded on I.

3.7 Suppose that $\lim_{x\to x_o} f(x) = l$. Prove the following:

(i) If $f(x) > 0$ in a neighbourhood of x_o then $l \geq 0$;

(ii) If $f(x) < 0$ in a neighbourhood of x_o then $l \leq 0$.

3.8 If $g(x) \leq f(x)$ and $\lim_{x\to x_o} g(x) = b$, $\lim_{x\to x_o} f(x) = a$, show that $b \leq a$.

3.9 Show that the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

has a limit only at the point $x = 1/2$.

3.10 Use the Cauchy definition of limit to prove the following:

(a) $\lim_{x\to 2}(x^2-1) = 3$ $\quad (b)$ $\lim_{x\to 3}(x^2-8x+7) = -8$

(c) $\lim_{x\to 0} x\sin\frac{2}{x} = 0$ $\quad (d)$ $\lim_{x\to\infty} \frac{x}{1+2x} = \frac{1}{2}$

(e) $\lim_{x\to\infty} \frac{2x^2-1}{3x^2+1} = \frac{2}{3}$ $\quad (f)$ $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$

3.11 Use the Heine definition of limit to prove the following:

(a) $\lim_{x\to 1}(x^2-1) = 0$ $\quad (b)$ $\lim_{x\to 0} x\sin\frac{1}{x} = 0.$

3.12 Find the limits:

(a) $\lim_{x\to 2} \frac{2x^2-3}{x-1}$ $\quad (b)$ $\lim_{x\to 0} \frac{x^3-1}{x-1}$

(c) $\lim_{x\to 0} \frac{\sin(2x)}{x}$ $\quad (d)$ $\lim_{x\to 0} \frac{1-\cos x}{x}$

(e) $\lim_{x\to 0} \frac{\tan x}{x}$ $\quad (f)$ $\lim_{x\to 3-} (x-[x]).$

3.13 Find the following limits, if they exist.

(a) $\lim_{x\to+\infty} \sin x$ $\quad (b)$ $\lim_{x\to+\infty} \cos x$ $\quad (c)$ $\lim_{x\to+\infty} \sin x^2$

3.14 Complete the following definitions.

$\lim_{x\to a} f(x) = l$	$\Longleftrightarrow$	$\forall \varepsilon > 0 \ \exists \delta > 0$	$(\lvert x-a\rvert < \delta$	$\Longrightarrow$	$\lvert f(x) - l\rvert < \varepsilon)$.
$\lim_{x\to a+} f(x) = l$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a-} f(x) = l$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a} f(x) = +\infty$	$\Longleftrightarrow$	$\forall M \ \exists \delta > 0$	$(\lvert x-a\rvert < \delta$	$\Longrightarrow$	$f(x) > M)$.
$\lim_{x\to a} f(x) = -\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a+} f(x) = +\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a+} f(x) = -\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a-} f(x) = +\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to a-} f(x) = -\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to \infty} f(x) = l$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to \infty} f(x) = +\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to \infty} f(x) = -\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to -\infty} f(x) = l$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to -\infty} f(x) = +\infty$	$\Longleftrightarrow$	....	....	....	....
$\lim_{x\to -\infty} f(x) = -\infty$	$\Longleftrightarrow$	$\forall M \ \exists D$	$(x < D$	$\Longrightarrow$	$f(x) < M)$.

3.15 Use the $\varepsilon - \delta$ definition of continuity to prove that each of the following functions is continuous at the specified point $x = x_o$:

(a) $f(x) = x^2, \quad x_o = 2$

(b) $f(x) = \cos x$, where x_o is a real number

(c) $f(x) = (x-1)^2, \quad x_o = 3$

(d) $f(x) = \sqrt{(2+x)}, \quad x_o = 2$

(e) $$f(x) = \begin{cases} \dfrac{x^2-3}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3, \end{cases}$$

$x_o = 2, \quad x_o = 3.$

3.16 Prove the following theorem.

Theorem 3.11 *Let $f(x)$ be defined on a given interval I.*

(i) *If f is continuous at $x_o \in I$ and $f(x_o) > 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o - \delta < x < x_o + \delta \quad \Longrightarrow \quad f(x) > 0).$$

(ii) *If f is continuous at $x_o \in I$ and $f(x_o) < 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o - \delta < x < x_o + \delta \quad \Longrightarrow \quad f(x) < 0).$$

(iii) *If f is right-continuous at $x_o \in I$ and $f(x_o) > 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o \le x < x_o + \delta \quad \Longrightarrow \quad f(x) > 0).$$

(iv) *If f is right-continuous at $x_o \in I$ and $f(x_o) < 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o \le x < x_o + \delta \quad \Longrightarrow \quad f(x) < 0).$$

(v) *If f is left-continuous at $x_o \in I$ and $f(x_o) > 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o - \delta < x \le x_o \quad \Longrightarrow \quad f(x) > 0).$$

(vi) *If f is left-continuous at $x_o \in I$ and $f(x_o) < 0$ then*

$$\exists \delta > 0 \ \ \forall x \in I \quad (x_o - \delta < x \le x_o \quad \Longrightarrow \quad f(x) < 0).$$

3.17 For each of the following functions find all points x of discontinuity.

(a) $\quad f(x) = \dfrac{x^2}{x^3 - 1}$ $\qquad$ (b) $\quad f(x) = \dfrac{1}{\sin x}$

(c) $\quad f(x) = \begin{cases} x, & x \text{ is rational} \\ 1 - x, & x \text{ is irrational} \end{cases}$ $\qquad$ (d) $\quad f(x) = \begin{cases} \log|x|, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Chapter 4

Real-Valued Functions II

4.1 Properties of Continuous Functions

4.1.1 Boundedness of Continuous Functions on Closed Intervals

Theorem 4.1 *If f is continuous on the closed interval $I = [a, b]$ then f is bounded on $[a, b]$.*

Proof. Without loss of generality, we may assume that the function $f(x)$ is not bounded above on the closed interval $[a, b]$. This means that for every real number we may choose as a bound, there exists at least one point x in $[a, b]$ such that $f(x)$ exceeds this bound. In particular, for every natural number n there exists a point x_n in $[a, b]$ such that

$$|f(x_n)| > n, \quad n = 1, 2, \ldots \tag{4.1}$$

The sequence $\{x_n\}$ so constructed is bounded $(a \le x_n \le b, \quad n = 1, 2, \ldots)$, so that it has a convergent subsequence $\{x_{n_k}\}$. Let $\lim_{k\to\infty} x_{n_k} = x_o$. Since all the terms of the sequence $\{x_{n_k}\}$ are in $[a, b]$, a closed interval, we conclude that $x_o \in [a, b]$ (Lemma 2.2). By assumption, $f(x)$ is continuous at the point $x = x_o$, which implies that

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_o).$$

But this is a contradiction to the claim (4.1) which implies that

$$\lim_{k\to\infty} f(x_{n_k}) = \infty. \quad \blacksquare$$

4.1.2 Extreme Value Theorem

Theorem 4.2 Extreme Value Theorem
If f is continuous on $[a, b]$ then there exist points $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \le f(x) \le f(x_2) \quad \textit{for all} \quad x \in [a, b];$$

that is,

$$f(x_1) = \min_{x\in[a,b]} f(x), \qquad f(x_2) = \max_{x\in[a,b]} f(x).$$

Proof.
Let A be the set of values of the function $f(x)$ when $a \le x \le b$:

$$A = f([a,b]) = \{f(x)|\ \ a \le x \le b\}.$$

By theorem 4.1, f is bounded on $[a,b]$, so that the set A is bounded both above and below. Let

$$m = \inf A \quad \text{and} \quad M = \sup A.$$

We shall show (by contradiction) that m and M are both values of the function f, that is, there are values x_1 and x_2, $x_1, x_2 \in [a,b]$, such that $f(x_1) = m$ and $f(x_2) = M$.

Firstly, suppose that there is no value of x in the interval $[a,b]$, for which $f(x) = M$, so that $M - f(x) > 0$ for all $x \in [a,b]$. Let

$$g(x) = \frac{1}{M - f(x)}, \quad x \in [a,b].$$

Since $f(x)$ is continuous on $[a,b]$, by Theorem 3.8, $g(x)$ is continuous on $[a,b]$, too. Hence, by theorem 4.1, $g(x)$ is bounded on $[a,b]$, so that there exists a number K such that $0 < |g(x)| \le K$ for every $x \in [a,b]$.

Now, since $M - f(x) > 0$ for all $x \in [a,b]$,

$$g(x) \le K \quad \Longrightarrow \quad \frac{1}{M - f(x)} \le K \quad \Longrightarrow \quad \frac{1}{K} \le M - f(x) \quad \Longrightarrow \quad f(x) \le M - \frac{1}{K},$$

for $x \in [a,b]$. This contradicts the fact that $M = \sup_{x\in[a,b]} f(x)$, the least upper bound for f on the interval $[a,b]$. Thus, $f(x)$ attains its supremum.

Finally, to prove that $y = f(x)$ attains its infimum m, observe that the supremum of $-f(x)$ is the infimum of $f(x)$, $x \in [a,\ b]$. ∎

4.1.3 Continuity and Order

Lemma 4.1 *Suppose that $g(x)$ is continuous at $x = x_o$.*

(a) *If $g(x_o) > 0$ then there exist $\delta > 0$ and $c > 0$ such that $g(x) > c$ for all x satisfying $|x - x_o| < \delta$.*

(b) *If $g(x_o) < 0$ then there exist $\delta > 0$ and $c > 0$ such that $g(x) < -c$ for all x satisfying $|x - x_o| < \delta$.*

Proof. Since $g(x)$ is a continuous function at the point $x = x_o$,

$$\forall \varepsilon > 0\ \ \exists \delta > 0\ \ (|x - x_o| < \delta \quad \Longrightarrow \quad -\varepsilon < g(x) - g(x_o) < \varepsilon).$$

(a) Since $g(x_o) > 0$ we can take $\varepsilon = \frac{1}{2}g(x_o)$. Thus, there is some $\delta > 0$ such that

$$|x - x_o| < \delta \quad \Longrightarrow \quad -\frac{1}{2}g(x_o) < g(x) - g(x_o) < \frac{1}{2}g(x_o).$$

This implies that $g(x) > c$ for all x satisfying $|x - x_o| < \delta$, where $c = \frac{1}{2}g(x_o)$.

(b) This follows from **(a)**, by considering $-g$. ∎

4.1.4 Intermediate Value Theorem

Theorem 4.3 *Let f be continuous on an interval I and let $[a, b] \subset I$. For every real number λ between $f(a)$ and $f(b)$ there exists a number c, $a < c < b$, such that*

$$f(c) = \lambda.$$

Proof. There are two possible cases:

$$(\mathbf{a}) \quad f(a) < \lambda < f(b), \qquad (\mathbf{b}) \quad f(b) < \lambda < f(a).$$

We will give a detailed proof for the first case. The corresponding proof for case **(b)** is similar. It also follows from **(a)** by considering the function $g = -f$.

Consider the following function

$$g(x) = f(x) - \lambda, \quad x \in [a, b].$$

We can see that:

$$\begin{aligned} &(i) \quad f(x) = \lambda \iff g(x) = 0, \\ &(ii) \quad g(a) = f(a) - \lambda < 0, \\ &(iii) \quad g(b) = f(b) - \lambda > 0. \end{aligned}$$

Define A to be the set of points of $[a, b]$ such that the function g is negative on the subinterval $[a, x]$:

$$A = \{x \mid a \leq x \leq b \ \& \ g \ \text{ is negative on } \ [a, x]\}.$$

The set A is non-empty, since $g(a) < 0$ which implies that $a \in A$. The set A is bounded above by the number b. Hence, by the axiom of completeness, A possesses a least upper bound. Let

$$c = \sup A.$$

Since $g(x)$ is continuous on $[a, b]$,

$$\begin{aligned} g(a) < 0 \quad &\Longrightarrow \quad \exists \delta_1 > 0 \ \ g(x) < 0 \ \text{ for } \ a \leq x < a + \delta_1, \\ g(b) > 0 \quad &\Longrightarrow \quad \exists \delta_2 > 0 \ \ g(x) > 0 \ \text{ for } \ b - \delta_2 < x \leq b. \end{aligned}$$

Therefore $a + \delta_1 \leq c$, $b - \delta_2 \geq c$, and we have

$$a < a + \delta_1 \leq c \leq b - \delta_2 < b \quad \Longrightarrow \quad a < c < b.$$

We now show that $g(c)$ is neither negative nor positive, that is the possibilities

$$\mathbf{1}. \ \ g(c) < 0, \qquad \mathbf{2}. \ \ g(c) > 0,$$

are both impossible.

Case 1. Suppose $g(c) < 0$. Then, since f is continuous, $g(x) < 0$ for x such that $c - \delta_3 < x < c + \delta_3$ for some $\delta_3 > 0$. In particular, $g(x) < 0$ for each x in the interval $c < x < c + \delta_3$. Let x be the midpoint of the interval $(c, c + \delta_3)$. Then $g(x) = g(c + \frac{1}{2}\delta_3) < 0$. But this contradicts the fact that $c = \sup A$, the least upper bound of the interval $[a, x]$ at which g is negative.

Case 2. Suppose now that $g(c) > 0$. Then $g(x) > 0$ for x in $c - \delta_4 < x < c + \delta_4$ for some $\delta_4 > 0$. In particular, $g(x) > 0$ for $c - \delta_4 < x < c$ and $g(c - \frac{1}{2}\delta_4) > 0$, again contradicting the fact that $c = \sup A$.

Therefore the only conclusion that can be drawn is that $g(c) = 0$, which implies that $f(c) = \lambda$, as required. ∎

An interval I in $\mathbb{R}$ is a set characterized by the property that if x, y are in I and $x < z < y$, then z is also in I. Consideration of cases $[a, b)$, (a, b), $(-\infty, a]$, etc., will easily convince the reader of the correctness of the characterization.

We shall now show that the continuous image of an interval is an interval.

Theorem 4.4 *Let f be defined on an interval I and continuous. Then the image*

$$f[I] = \{f(x) \mid x \in I\}$$

is also an interval.

Proof. Let $J = f[I]$. To show that J is an interval, consider $u < w < v$, where $u,\ v \in J$ and show that $w \in J$. Since $u,\ v \in J$, there are points $x,\ y$ in I such that $f(x) = u,\ \ f(y) = v$. Thus $x \neq y$. We may assume, without loss of generality that $y < x$. By the Intermediate Value Theorem, there exists $z,\ \ y < z < x$, such that $f(z) = w$. Thus $w \in J$, as required. ∎

Example 4.1 *Illustrating the concept of the image of an interval.*

Let $f(x) = x^2, \quad -1 < x < 1$. Then:

$$f[\,(-1,\ 1)\,] = [0, 1)$$

$$f[\,(-\frac{1}{2}, 1)\,] = [0, 1)$$

$$f[\,(-\frac{1}{2}, \frac{1}{4}]\,] = [0, \frac{1}{4})$$

$$f[\,[-\frac{1}{2}, \frac{1}{4})\,] = [0, \frac{1}{4}]$$

∎

4.1.5 The Fixed Point Theorem of Banach

Theorem 4.5 Fixed Point Theorem

Suppose that f is a continuous function defined on a closed interval $I = [a, b]$ that maps I into I and has the property

$$|f(x) - f(y)| \leq \alpha |x - y| \tag{4.2}$$

for all $x, y \in I$ with $0 < \alpha < 1$.

(i) *Then the function $f(x)$ has exactly one fixed point, that is, there is exactly one point $x_o \in I$ such that $f(x_o) = x_o$.*

(ii) *If x_1 is any point of I and $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$, then*

$$\lim_{n\to\infty} x_n = x_o.$$

Proof.

(i) To establish the uniqueness of the fixed point x_o, suppose that there is another point x' such that $f(x') = x'$. Using (4.2), we find that

$$|x_o - x'| = |f(x_o) - f(x')| \leq \alpha |x_o - x'|.$$

Since $0 < \alpha < 1$, it is clear that $|x_o - x'| = 0$ which implies that $x' = x_o$.

(ii) Let x_1 be any point of I and let $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$. Then

$$|x_2 - x_3| = |f(x_1) - f(x_2)| \leq \alpha |x_1 - x_2|,$$

$$|x_3 - x_4| = |f(x_2) - f(x_3)| \leq \alpha |x_2 - x_3| \leq \alpha^2 |x_1 - x_2|.$$

In general, we have

$$|x_n - x_{n+1}| \leq \alpha^{n-1} |x_1 - x_2|.$$

Let $m \geq n$ be any natural numbers. Then, using the triangle inequality, we find that

$$\begin{aligned}
|x_n - x_m| &= |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{m-1} - x_m)| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\leq |x_1 - x_2|(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^{m-2}) \\
&= (1 + \alpha + \cdots + \alpha^{m-n-1})\alpha^{n-1}|x_1 - x_2| \\
&= |x_1 - x_2|\alpha^{n-1} \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \\
&< \frac{|x_1 - x_2|}{1 - \alpha} \cdot \alpha^{n-1}.
\end{aligned}$$

Since $|x_n - x_m| \le \frac{|x_1-x_2|}{1-\alpha}\alpha^{n-1}$ and $\lim\limits_{n\to\infty} \dfrac{|x_1 - x_2|}{1-\alpha}\alpha^{n-1} = 0$, we conclude that for every $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x_m| < \varepsilon \quad \text{for} \quad n, m > N.$$

This means that $\{x_n\}$ is a Cauchy sequence, so that it converges. Let

$$\lim_{n\to\infty} x_n = x_o.$$

and consider now the sequence $\{f(x_n)\}$. Since f is continuous, $\lim\limits_{n\to\infty} f(x_n) = f(x_o)$. Now,

$$f(x_n) = x_{n+1} \quad \Longrightarrow \quad f(x_o) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x_o.$$

Therefore we have established that $f(x_o) = x_o$ and the proof is completed. ∎

4.2 Uniform Continuity

Uniform continuity is a global property of a function on a set, whereas continuity is a local property. We recall that a function f with domain D is continuous at a point $x_o \in D$ if for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that for all $x \in D$,

$$|x - x_o| < \delta \quad \Longrightarrow \quad |f(x) - f(x_o)| < \varepsilon. \tag{4.3}$$

We note that the value of δ in (4.3) depends on the choice of ε and in addition may depend on x_o.

Now, if (4.3) is satisfied for every x_o in a given subset I of the domain D of f, then f is continuous on the set I. Each x_o gives us a value of δ associated with it. When it is possible to obtain one number $\delta > 0$ which will satisfy (4.3) for all points $x_o \in I$, then we say that $f(x)$ is **uniformly continuous** on I.

To contrast the difference between continuity on a set I and uniform continuity on I, we shall, firstly rewrite (4.3) as:

$$\forall\varepsilon > 0 \;\; \forall x_1 \in I \;\; \exists\delta > 0 \quad \forall x_2 \in I \quad (|x_1 - x_2| < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon).$$

Uniform Continuity expresses the fact that δ depends only on ε and does not depend on $x_1,\ x_2$ in D.

Definition 4.1 *The function f with domain D is* **uniformly continuous** *on the set I, $I \subset D$, if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1,\ x_2 \in I \quad (|x_1 - x_2| < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon)\,. \tag{4.4}$$

Example 4.2 *Show that the function*

$$f(x) = \sin x$$

is uniformly continuous on the set of all real numbers $\mathbb{R} = (-\infty, \infty)$.

Solution. Let $\varepsilon > 0$ be given and let $x_1,\ x_2$ be any pair of real numbers. We have

$$\begin{aligned} |f(x_1) - f(x_2)| &= |\sin x_1 - \sin x_2| \\ &= 2\left|\sin \frac{x_1 - x_2}{2} \cos \frac{x_1 + x_2}{2}\right| \\ &\leq 2\left|\frac{x_1 - x_2}{2}\right| \cdot 1 \\ &= |x_1 - x_2|. \end{aligned}$$

If $\delta = \varepsilon$, therefore,

$$|x_1 - x_2| < \delta = \varepsilon \quad \Longrightarrow \quad |f(x_1) - f(x_2)| \leq |x_1 - x_2| < \varepsilon. \quad \blacksquare$$

Example 4.3 *Show that the function $f(x) = \dfrac{1}{x}$ is not uniformly continuous on the open interval $I = (0, 1)$.*

Solution. Let $x_1 = \dfrac{1}{n}$ and $x_2 = \dfrac{1}{2n}$, where n is a natural number. Then

$$|x_1 - x_2| = \frac{1}{2n}$$

and

$$|f(x_1) - f(x_2)| = |\frac{1}{x_1} - \frac{1}{x_2}| = |n - 2n| = n \geq 1.$$

We can see, therefore, that there exists $\varepsilon > 0$ such that for every $\delta > 0$ there are two points x_1 and x_2 for which

$$|x_1 - x_2| < \delta \quad \text{but} \quad |f(x_1) - f(x_2)| > \varepsilon.$$

Namely, let $\varepsilon = 1$. Then, given $\delta > 0$, choose any $n > \frac{1}{2\delta}$. If $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{2n}$ then we have

$$|x_1 - x_2| = \frac{1}{2n} \quad \text{but} \quad |f(x_1) - f(x_2)| = n \geq 1 = \varepsilon.$$

Thus the negation of (4.4) holds which proves that $f(x) = \dfrac{1}{x}$ is not uniformly continuous on the interval $I = (0, 1)$.

Note, that $f(x) = \dfrac{1}{x}$ is continuous on every point $x_o \in I = (0, 1)$, so that $f(x)$ is continuous on I. ∎

Example 4.4 *Show that the function* $f(x) = \sin \dfrac{1}{x}$ *is not uniformly continuous on the interval* $I = (0, \dfrac{2}{\pi}]$.

Solution. Let

$$x_1 = \frac{2}{(2n+1)\pi}, \qquad x_2 = \frac{1}{n\pi},$$

where n is a natural number. Then

$$f(x_1) = \sin(2n+1)\frac{\pi}{2} = \pm 1, \quad f(x_2) = \sin n\pi = 0, \quad |f(x_1) - f(x_2)| = 1,$$

but $|x_1 - x_2| = \dfrac{1}{n(2n+1)\pi}$, when n increases, can be made sufficiently small.

With $\varepsilon = 1$ there is no δ that is suitable for all x in $(0, \dfrac{2}{\pi}]$. Hence the negation of (4.4) holds, which implies that the function $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on the interval $I = \left(0, \frac{2}{\pi}\right]$. ∎

It can be shown, however, that $f(x) = \sin \frac{1}{x}$ is uniformly continuous on any closed interval that does not contain 0.

Example 4.5 *Show that* $f(x) = \sin \frac{1}{x}$ *is uniformly continuous on the interval* $I = [c, \infty)$, *where* $c > 0$.

Solution. If $x_1,\ x_2 \in [c, \infty)$ and $|x_1 - x_2| < \delta = c^2\varepsilon$, then, since $|\sin a - \sin b| \leq |b - a|$,

$$|f(x_1) - f(x_2)| \leq \frac{1}{x_1}\,\frac{1}{x_2}\,|x_1 - x_2| < \frac{1}{c}\,\frac{1}{c}\,c^2\varepsilon = \varepsilon. \quad ∎$$

We note that the concept of uniform continuity is stronger than the concept of continuity in the sense that every uniformly continuous function on a given set I is automatically continuous on I. We have seen in examples 4.3 and 4.4 some continuous functions that are not uniformly continuous.

Now we will prove that if I is a closed interval then every continuous function on I is also uniformly continuous on I.

Theorem 4.6 *If f is a continuous function on a closed interval $I = [a, b]$, then f is also uniformly continuous on I.*

Proof. (by contradiction)
Suppose that f is continuous on $I = [a, b]$ but not uniformly continuous on I. Then the negation of (4.4) is true, which implies that there exists an $\varepsilon_o > 0$ such that for every $\delta > 0$ there are $x,\ y \in I$ that satisfy

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon_o.$$

Let $\delta = 1/n$, where n is a positive integer. For this particular value of δ, we can find two numbers $x_n,\ y_n \in I$ that satisfy

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_o.$$

We can, therefore, construct two sequences $\{x_n\}$ and $\{y_n\}$ with terms in I such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_o \quad \text{for} \quad n = 1, 2, \ldots. \tag{4.5}$$

We note that both sequences $\{x_n\}$ and $\{y_n\}$ are bounded, as

$$x_n,\ y_n \in I = [a, b] \text{ for } n = 1, 2, \ldots.$$

Thus, by the Bolzano-Weierstrass theorem, the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges to $c \in I$. Now $n_1 < n_2 < \cdots$ determines a subsequence $\{y_{n_k}\}$ of $\{y_n\}$. Since

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c|,$$

it is clear that $\{y_{n_k}\}$ also converges to c. Now, f is continuous at the point c, so that

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(c).$$

But this is not possible since (4.5) implies that $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_o$ for all k.

The above contradiction implies that the assumption we made is false. Hence $f(x)$ is uniformly continuous on I. ∎

4.3 Continuity of Inverse Functions

Continuous functions defined on intervals (a, b) which have inverses are rather special — they must be either strictly increasing throughout the interval (a, b), or strictly decreasing throughout the interval. This is the result we shall now prove.

Theorem 4.7 *Let f be a continuous function defined on $I = (a, b)$ and suppose f has an inverse. Then f is a monotone function on I.*

Proof. Suppose f is not monotone decreasing. Then there are points c, d in I such that $c < d$ and $f(c) < f(d)$.

We shall prove that, if x is such that $c < x < d$, then $f(c) < f(x) < f(d)$. Suppose not. Then there is $x \in (c, d)$ for which $f(x) < f(c)$ or $f(x) > f(d)$. In the first case, we have $f(x) < f(c) < f(d)$, so there must be $y \in (x, d)$ such that $f(y) = f(c)$, by the Intermediate Value Theorem. This would contradict the fact that f is one to one, since $y \neq c$. Similarly, it cannot happen that $f(x) > f(d)$. Hence

$$c < x < d \quad \Longrightarrow \quad f(c) < f(x) < f(d). \tag{4.6}$$

Suppose $u < c$. We shall show that $f(u) < f(c)$. Suppose not, then $f(u) > f(c)$. If $f(d) < f(u)$, then $f(c) < f(d) < f(u)$, so there is v in (u, c) such that $f(v) = f(d)$, a contradiction. Thus $f(u) < f(d)$. But then $f(c) < f(u) < f(d)$, so there is w in (c, d) such that $f(u) = f(w)$, again a contradiction. Hence

$$u < c \quad \Longrightarrow \quad f(u) < f(c). \tag{4.7}$$

Analogously, one has

$$d < t \quad \Longrightarrow \quad f(d) < f(t). \tag{4.8}$$

It follows from (4.6), (4.7), and (4.8) that f is monotone increasing on (a, b). ∎

In fact, the inverse function of a continuous function defined on an interval I will itself be continuous.

Theorem 4.8 *Let f be a continuous function defined on an interval I, with an inverse function f^{-1} defined on J, the image of I under f. Then f^{-1} is a continuous function on J.*

Proof. By above, we may assume that f is monotone increasing on I. Suppose f^{-1} is not continuous at $y_o \in J$. Then there is $\varepsilon > 0$ and $\{y_n\} \subset J$ such that $\lim_{n\to\infty} y_n = y_o$ and

$$\left| f^{-1}(y_n) - f^{-1}(y_o) \right| \geq \varepsilon.$$

Without loss of generality we may assume that $\{y_n\}$ is monotone, say monotone decreasing to y_o, since every convergent sequence in $\mathbb{R}$ is bounded, and, hence, has a monotone subsequence. But then $\{x_n\}$, where $x_n = f^{-1}(y_n)$, will be a monotone decreasing sequence in I bounded below by x_o. Hence it will converge to u, say, where $u \geq x_o$. Hence $\lim_{n\to\infty} f(x_n) = f(u)$, by continuity of f at $x = u$.

But $f(x_n) = y_n$ and $\lim_{n\to\infty} y_n = y_o$. Hence $f(u) = y_o$. Since $f(x_o) = y_o$, we conclude that $u = x_o$. But then $\{x_n\}$ is a sequence which converges to x_o, contradicting

$$|x_n - x_o| = \left|f^{-1}(y_n) - f^{-1}(y_o)\right| \geq \varepsilon.$$

We have shown that f^{-1} is continuous at all y in J, as required. ∎

4.4 Functions of Two Variables

Definition 4.2 *Let $\mathcal{D}$ be a subset of $\mathbb{R}^2$ and let F be a real-valued function defined on $\mathcal{D}$, so that to each point $(x, y) \in \mathcal{D}$ there is assigned a unique real number denoted by $F(x, y)$.*

The set $\mathcal{D}$ is called the **domain** *of F. The set $\mathcal{V}$ of all possible values of the function F,*

$$\mathcal{V} = \{F(x, y) : \ (x, y) \in \mathcal{D}\},$$

is called the **range** *of F.*

$$F(x,y) = \sqrt{4 - x^2 - y^2}$$

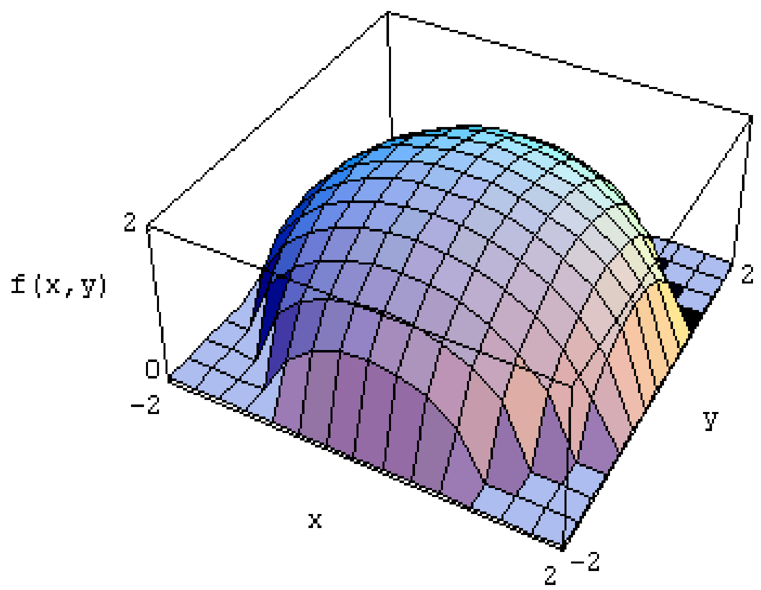


Figure 4.1: Three-dimensional graph of the function $F(x, y)$.

Example 4.6 *Find the domain and range of the function*

$$F(x,y) = \sqrt{4 - x^2 - y^2}.$$

Solution. The domain of definition of F is determined by the inequality $4-x^2-y^2 \geq 0$. Hence the domain of F is the set of all points on the circle $x^2+y^2 = 4$ and all points interior to that circle:

$$\mathcal{D} = \{(x,y): \ \ x^2 + y^2 \leq 4\}.$$

The set of possible values of F is the interval $[0, 2]$,

$$\mathcal{V} = [0, 2],$$

since $0 \leq 4 - x^2 - y^2 \leq 4 \quad \Longrightarrow \quad 0 \leq F(x,y) \leq 2$. In Figure 4.1, a two-dimensional representation of the three-dimensional graph of the function F is given. ∎

4.4.1 Limits of Functions of Two Variables

We define the distance between two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ of $\mathbb{R}^2$ as

$$|z_1 - z_2| = |(x_1, y_1) - (x_2, y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Let $(x_o, y_o) \in \mathbb{R}^2$ and let $\delta > 0$ be given. The inequality

$$|(x, y) - (x_o, y_o)| < \delta$$

determines an open disk, namely the set

$$\{(x, y): \ (x - x_o)^2 + (y - y_o)^2 < \delta^2\}$$

of all points interior to the circle with radius δ centred at the point (x_o, y_o), which is called *a δ-neighbourhood* of the point (x_o, y_o).

The inequality

$$0 < |(x, y) - (x_o, y_o)| < \delta$$

represents *a δ-neighbourhood* of (x_o, y_o) with the centre (x_o, y_o) deleted, which is called *a deleted δ-neighbourhood* of the point (x_o, y_o).

Suppose that the function $F(x, y)$ is defined in a neighbourhood of a given point (x_o, y_o), but not necessarily at (x_o, y_o). We say that $F(x, y)$ has a limit L as (x, y) approaches (x_o, y_o), written $\lim\limits_{(x,y)\to(x_o,y_o)} F(x, y)$, if the difference between L and the values of the function F are arbitrarily small for all points (x, y) sufficiently close to (x_o, y_o).

Definition 4.3 *(Limit of a function of two variables)*
A function F defined in a deleted neighbourhood of the point (x_o, y_o) is said to have the **limit** *L if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|F(x, y) - L| < \varepsilon$ for all points (x, y) in the deleted δ-neighbourhood of the point (x_o, y_o):*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \quad (0 < |(x, y) - (x_o, y_o)| < \delta \quad \Longrightarrow \quad |F(x, y) - L| < \varepsilon).$$

When the limit is L, we shall write $\lim\limits_{(x,y)\to(x_o,y_o)} F(x, y) = L$.

Example 4.7 *Let $F(x, y) = (x + y)\sin(x + y)$, $\mathcal{D} = \mathbb{R}^2$. Show that*

$$\lim_{(x,y)\to(0,0)} F(x, y) = 0$$

.

Solution. Let $\varepsilon > 0$ be given. We are to find a $\delta > 0$ such that

$$|(x, y) - (0, 0)| = \sqrt{x^2 + y^2} < \delta \quad \Longrightarrow \quad |F(x, y) - 0| < \varepsilon.$$

We have

$$|F(x, y)| = |(x + y)\sin(x + y)| \le |x + y| \le |x| + |y|.$$

Now, $\sqrt{x^2 + y^2} < \delta$ implies that $|x| < \delta$ and $|y| < \delta$. It is clear that if we choose $\delta = \frac{\varepsilon}{2}$, then

$$\sqrt{x^2 + y^2} < \delta \quad \Longrightarrow \quad |F(x, y)| \le |x| + |y| < 2\delta = \varepsilon. \ \blacksquare$$

4.4.2 Continuity of a Function of Two Variables

Definition 4.4 *A function $F(x,y)$ defined at every point (x,y) in a neighbourhood of a given point (x_o,y_o) is said to be* **continuous** *at (x_o,y_o), if $\lim\limits_{(x,y)\to(x_o,y_o)} F(x,y)$ exists and*

$$\lim_{(x,y)\to(x_o,y_o)} F(x,y) = F(x_o,y_o).$$

We note that if $F(x,y)$ is continuous at (x_o,y_o), this requires that $F(x,y)\to F(x_o,y_o)$ as $(x,y)\to(x_o,y_o)$ by any path in a neighbourhood of the point (x_o,y_o).

Consider now $F(x,y)$ as a function of the variable x with the value of y fixed, $y=y_o$. If $F(x,y)$ is continuous at the point (x_o,y_o), then clearly $F(x,y_o)$, as a function of one variable x is continuous at the point $x=x_o$. This only requires that $F(x,y_o)\to F(x_o,y_o)$ as $x\to x_o$, or $(x,y)\to(x_o,y_o)$ along the horizontal line $y=y_o$. Similarly, $F(x_o,y)$ is continuous with respect to y at the point $y=y_o$.

Hence, continuity in (x,y) implies continuity in each variable separately, but the converse is not true, as shown in the Exercises.

4.4.3 The Implicit Function Theorem

The theorem says that the equation $F(x,y)=0$ defines uniquely a function $y=f(x)$ in a neighbourhood of the point (x_o,y_o), provided that $F(x,y)$ satisfies some conditions, in particular $F(x_o,y_o)=0$.

Theorem 4.9 The Implicit Function Theorem I

Let F be a function of two variables x and y, where $x \in I,\ \ y \in J$, I and J are intervals in $\mathbb{R}$. Suppose that, for $(x, y) \in \mathcal{D} = I \times J$:

1. *$F(x, y)$ is continuous;*
2. *$F(x_o, y_o) = 0$ for some point (x_o, y_o), where $x_o \in I,\ \ y_o \in J$;*
3. *$F(x, y)$ is monotonic as a function of $y \in J$ for each fixed value of $x \in I$.*

Then there are positive numbers h and k that define the rectangle

$$\mathcal{R} = \{(x, y) : |x - x_o| < h,\ \ |y - y_o| < k\}$$

such that the equation

$$F(x, y) = 0$$

defines y as a continuous function of x, $y = f(x)$, for $x \in I_o = \{x : |x - x_o| < h\} \subset I$, with range contained in the interval $J_o = \{y : |y - y_o| < k\} \subset J$. Moreover

$$f(x_o) = y_o.$$

Refer to Figure 4.2. The theorem says that for each x in $I_o = \{x : |x - x_o| < h\}$ there is a unique number y in $J_o = \{y : |y - y_o| < k\}$ which satisfies the equation $F(x, y) = 0$. The totality of points $(x, y) \subset \mathcal{R}$ for which $F(x, y) = 0$ determines a function f whose domain is the interval I_o and whose range is contained in J_o. The function f defined in this way is a continuous function of x on its domain I_o and assumes the value y_o at the point x_o.

(a) $F(x, y) = 0 \implies f(x_o) = y_o$ **(b)** $\exists (x^*, y^*) \in I_o \times J_o,\ \ F(x^*, y^*) = 0$

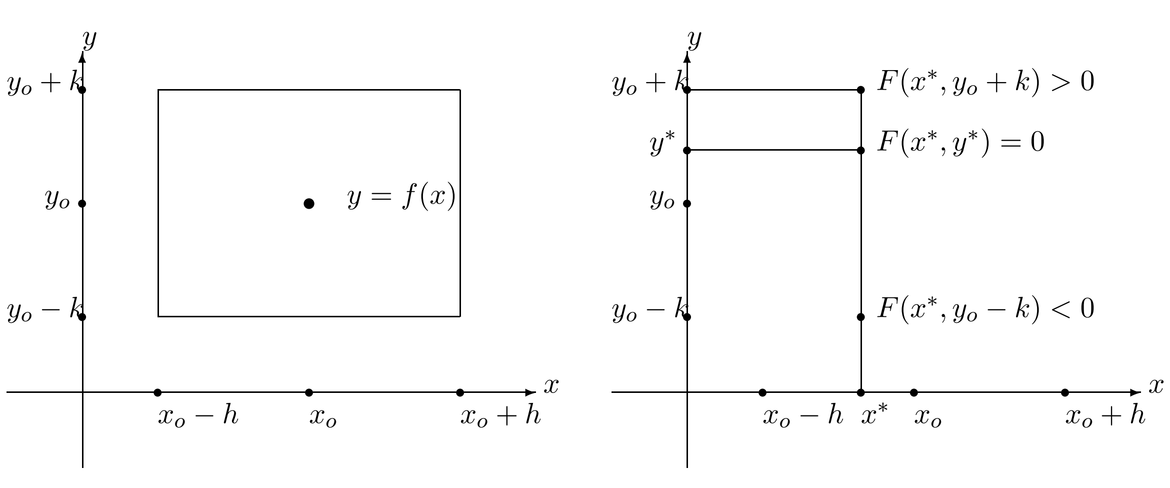


Figure 4.2: Illustrating the Implicit Function Theorem I.

Proof. We shall give a detailed proof for the case where $F(x,y)$ is an increasing function of y for each fixed value of x, $(x,y) \in \mathcal{D}$, the domain of F. The proof for the other case is left to the student as an exercise.

The proof consists of several steps:

1. We show that there is $k > 0$ such that
$$F(x_o, y_o - k) < 0 \text{ and } F(x_o, y_o + k) > 0.$$
2. We show that there is $h > 0$ such that
$$F(x, y_o - k) < 0 \quad \text{and} \quad F(x, y_o + k) > 0 \quad \text{for} \quad |x - x_o| < h.$$
3. We show that, for each value of x in the interval $I_o = (x_o - h, x_o + h)$ there is a unique number $y = y^* \in J_o = (y_o - k, y_o + k)$ such that $F(x, y^*) = 0$.
4. We show that the function $y = f(x)$ determined in step 3 satisfies the condition $f(x_o) = y_o$.
5. We show that the function f is continuous at the point $x = x_o$.
6. We complete the proof by showing that the function f is continuous at x_1, where x_1 is any point of the interval I_o.

Step 1. Let $x = x_o$. Then $F(x_o, y)$ can be considered as a function of y, $y \in J$. By hypothesis 2, $F(x_o, y_o) = 0$. By hypothesis 3, $F(x_o, y)$ is increasing. Hence, if $y \in J$, then

$$y < y_o \implies F(x_o, y) < 0, \tag{4.9}$$

$$y > y_o \implies F(x_o, y) > 0. \tag{4.10}$$

There is a positive value k such that the square

$$S = \{(x,y) : |x - x_o| \le k, \ \ |y - y_o| \le k\}$$

is contained in the domain $\mathcal{D}$ of the function $F(x,y)$. Thus, the inequalities (4.9) and (4.10) imply that
$$F(x_o, y_o - k) < 0 \text{ and } F(x_o, y_o + k) > 0.$$

Step 2. The function $F(x, y_o - k)$ is continuous as a function of x, so that the inequality $F(x_o, y_o - k) < 0$ implies that $F(x, y_o - k) < 0$ in a neighbourhood of the point x_o. Similarly we conclude that $F(x, y_o + k) > 0$ in a neighbourhood of x_o. Therefore, there is a (sufficiently small) number h, $0 < h \le k$, such that

$$F(x, y_o - k) < 0 \quad \text{for} \quad |x - x_o| < h \tag{4.11}$$

and
$$F(x, y_o + k) > 0 \quad \text{for} \quad |x - x_o| < h \tag{4.12}$$

Step 3. Let $x = x^*$ be any value of x in the interval $I_o = (x_o - h, x_o + h)$. We have: $F(x^*, y_o - k) < 0$, $F(x^*, y_o + k) > 0$. By hypothesis 1, the function $F(x^*, y)$ is a continuous function of y. By the Intermediate Value Theorem, therefore, there is y^*, $y_o - k < y^* < y_o + k$, such that $F(x^*, y^*) = 0$. See the diagram in Figure 4.2(b).

By hypothesis 3, if $y \in J_o = (y_o - k, y_o + k)$, then

$$y < y^* \implies F(x^*, y) < 0,$$

$$y > y^* \implies F(x^*, y) > 0.$$

This implies that $y = y^*$ is the only value of y in the interval J_o for which $F(x^*, y^*) = 0$.

We have shown that for every $x = x^*$ in the interval $I_o = (x_o - h, x_o + h)$ there is a unique value $y = y^*$ in the interval J_o such that $F(x^*, y^*) = 0$. Hence, the equation $F(x, y) = 0$ defines a function in the rectangle $\mathcal{R} = I_o \times J_o$. Let $y = f(x)$ be the function under consideration.

Step 4. In particular, when $x^* = x_o$, we obtain

$$y^* = f(x^*) = y_o,$$

using hypothesis 2 and the uniqueness of y^* for any fixed value of $x \in I_o$.

Step 5. Let $\varepsilon > 0$ be given. Suppose that $\varepsilon < k$. Replace the square S in steps 1—3 of the proof by the square

$$S_\varepsilon = \{(x, y) : |x - x_o| \leq \varepsilon, \ \ |y - y_o| \leq \varepsilon\}.$$

We arrive at the conclusion that there is a value h', $h' < h$, such that $y = f(x)$ is a function on $I' = \{x : |x - x_o| < h'\}$ whose range is contained in

$$J' = \{y : |y - y_o| < \varepsilon\} = \{y : |f(x) - f(x_o)| < \varepsilon\}.$$

Therefore,

$$\forall 0 < \varepsilon < k \ \ \exists h' > 0 \ \ (|x - x_o| < h' \implies |f(x) - f(x_o)| < \varepsilon). \tag{4.13}$$

We have proved that (4.13) holds for any $\varepsilon < k$. Hence (4.13) is automatically true for every $\varepsilon \geq k$ and we conclude that f is continuous at the point x_o.

Step 6. To show that f is continuous on the interval I_o, we need to show that f is continuous at $x = x_1$, where x_1 is any point of I_o.

Let $x_1 \in I_o$ be given. Let $y_1 = f(x_1)$. Then the function $y = f(x)$ satisfies the condition

$$F(x_1, y_1) = 0$$

so that we can repeat steps 1 — 5 of the proof replacing (x_o, y_o) by (x_1, y_1) to arrive at the following conclusion:

There are positive numbers h_1 and k_1 such that the equation $F(x, y) = 0$ defines a function $y = f(x)$ for $x \in I_1 = \{x : |x - x_1| < h_1\} \subset I_o$ with range contained in $J_1 = \{y : |y - y_1| < k_1\} \subset J_o$. Moreover, the function $y = f_1(x)$ is continuous at the point $x = x_1$.

Since the function $y = f_1(x)$ is the only function defined on I_1 that satisfies $F(x, y) = 0$, we have $f(x) = f_1(x)$ for $x \in I_1$. Hence $f(x)$ is continuous at the point $x = x_1$. ∎

4.5 Exercises

4.1 Use the $\varepsilon - \delta$ definition of uniform continuity to prove that each of the following functions is uniformly continuous on the specified interval.

$(a)\quad f(x) = x^3 + 1, \quad x \in (1, 2) \qquad (b)\quad f(x) = \sqrt{1+x}, \quad x \in (-1, 2)$

$(c)\quad f(x) = \sin(3x), \quad x \in (-\infty, \infty) \quad (d)\quad f(x) = \frac{1}{x}, \quad x \in (1, \infty)$

$(e)\quad f(x) = \sqrt{x}, \quad x \in (0, \infty).$

4.2 Show that each of the following functions is not uniformly continuous on the specified interval.

$$(a)\quad f(x) = x^2, \quad x \in (-\infty, \infty)$$

$$(b)\quad f(x) = \frac{1}{x^2}, \quad x \in (0, 2).$$

4.3 Use the definition of limit of a function of two variables to prove that

$$\lim_{(x,y)\to(1,2)} (3x + 2y) = 7.$$

4.4 Let $F(x, y) = x^2 + y^2 - 1$.

(a) Determine y as a continuous function of x, given that:

$$(i)\quad y = -\frac{1}{\sqrt{2}} \quad \text{when} \quad x = \frac{1}{\sqrt{2}},$$

$$(ii)\quad F(x, y) = 0.$$

(b) Is it possible to determine y as a continuous function of x when the following hold?

$$(i)\quad x_o = 1 \quad \text{gives} \quad y_o = 0,$$

$$(ii)\quad F(x, y) = 0.$$

4.5 Discuss when it is possible to determine y as a continuous function of x, $0 < x < 1$, when x, y satisfy:

$$f(x, y) = x + y + \sin xy \equiv 0.$$

4.6 Let $F(x, y) = \dfrac{xy}{1 + x^2y^2}$.

(a) Determine the domain of F.

(b) Is $F(x, y)$ a continuous function? Justify your answer.

4.7 Consider the following function

$$F(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Show that for every x_o, $f_2(y) = F(x_o, y)$ is continuous.

(b) Show that for every y_o, $f_1(x) = F(x, y_o)$ is continuous.

(c) Show that $H = F(x,y)$ is not continuous at $(0,0)$.

(d) Show that $H = F(x,y)$ is a bounded function of x and y.

Chapter 5

Differentiation

5.1 Definition of Derivative

Geometrically, "$f(x)$ is differentiable at x_o" means that a tangent line can be drawn to the curve $y = f(x)$ at $x = x_o$. The slope of the tangent line is $f'(x_o)$. This geometric interpretation is shown in Figure 5.1, where α denotes the angle between the tangent line to the curve $y = f(x)$ at x_o and the positively directed x-axis. We note that

$$\frac{f(x_o + h) - f(x_o)}{h}$$

is the gradient of the line through the points $(x_o, f(x_o))$ and $(x_o + h, f(x_o + h))$. As $h \to 0$, the secant line approaches the position of the tangent line to the curve $y = f(x)$ at x_o. Refer to Figure 5.1 again to see that

$$\lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} = \tan \alpha.$$

Definition 5.1 *Let f be a function defined on a given interval $I \subset \mathbb{R}$ and let x_o be an interior point of I so that $f(x)$ is defined in a neighbourhood of the point x_o. If the limit*

$$\lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} \tag{5.1}$$

exists, then the function $f(x)$ is said to be **differentiable** *at the point $x = x_o$. The value of this limit is called the* **derivative** *of $f(x)$ at the point $x = x_o$.*

If the limit (5.1) does not exist, then it is said that the function $f(x)$ is not differentiable at $x = x_o$ or $f(x)$ does not possess a derivative at $x = x_o$.

Notations: The derivative of $y = f(x)$ is denoted in a variety of ways. The most commonly used notations for the derivative of $y = f(x)$ at $x = x_o$ are

$$\text{(i)}\quad f'(x_o), \qquad \text{(ii)}\quad \left.\frac{dy}{dx}\right|_{x=x_o}, \quad \text{(iii)}\quad \left.\frac{df}{dx}\right|_{x=x_o}, \quad \text{(iv)}\quad Df|_{x=x_o}.$$

When y is a function of t, $y = f(t)$, the derivative of y at $t = t_o$ is also denoted by $\dot{y}(t_o)$.

For a function $f(x)$ which is differentiable at the point $x = x_o$ we therefore have

$$f'(x_o) = \lim_{h\to 0} \frac{f(x_o + h) - f(x_o)}{h} = \lim_{x\to x_o} \frac{f(x) - f(x_o)}{x - x_o}.$$

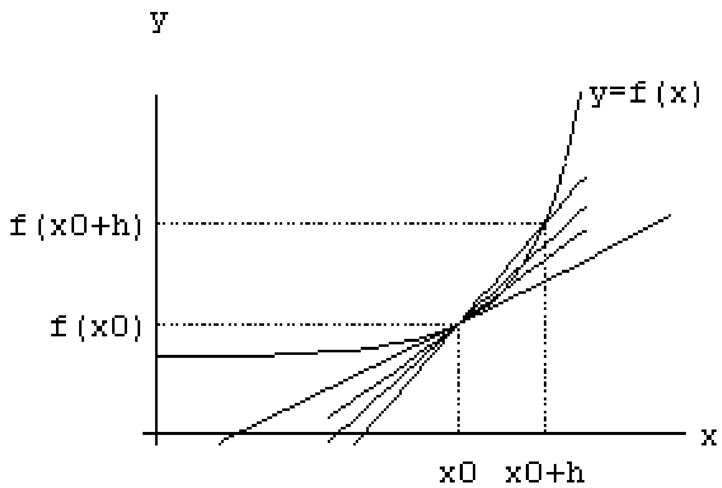


Figure 5.1: Illustrating the concept of the derivative $f'(x_o)$ of the function $y = f(x)$ at the point x_o.

From the definition, the derivative can be regarded as expressing a relationship between the increment in $y = f(x)$ corresponding to an increment h in x when $x = x_o$. It is formally stated in the following theorem.

Theorem 5.1 *Let f be defined on an interval I and let x_o be an interior point of I. f is differentiable at x_o if and only if there exists a constant A such that, for all h sufficiently small,*

$$f(x_o + h) = f(x_o) + A\,h + h\,\varepsilon(h),$$

where $\lim_{h\to 0} \varepsilon(h) = 0.$

Proof.

$\Longrightarrow$ Assume $f'(x_o)$ exists. Then

$$\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}=f'(x_o),\quad \text{so that}\quad \frac{f(x_o+h)-f(x_o)}{h}=f'(x_o)+\varepsilon(h),$$

where $\lim_{h\to 0}\varepsilon(h)=0$. Hence $f(x_o+h)=f(x_o)+Ah+h\,\varepsilon(h)$, where $A=f'(x_o)$.

$\Longleftarrow$ If $f(x_o+h)=f(x_o)+Ah+h\,\varepsilon(h)$ and $\lim_{h\to 0}\varepsilon(h)=0$, then

$$\frac{f(x_o+h)-f(x_o)}{h}=A+\varepsilon(h).$$

Hence $f'(x_o)=A$. The argument shows that A, if it exists, is unique and is equal to $f'(x_o)$. ∎

Example 5.1 *Show that the function $f(x)=2x^2+3$ is differentiable at the point $x_o=1$ and find the derivative $f'(1)$.*

Solution. If $x_o=1$, then

$$\begin{aligned}\frac{f(x_o+h)-f(x_o)}{h} &= \frac{2(1+h)^2+3-(2+3)}{h}\\ &= 4+2h.\end{aligned}$$

Since $\lim_{h\to 0}(4+2h)$ exists, we conclude that $f(x)$ is differentiable at $x_o=1$ and we have

$$f'(x_o)=f'(1)=\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}=\lim_{h\to 0}(4+2h)=4.$$

Thus $f'(1)=4$. ∎

Example 5.2 *Show that the function $f(x)=2x^2+3$ is differentiable at any point x_o.*

Solution. We have

$$\begin{aligned}f(x_o+h)-f(x_o) &= 2(x_o+h)^2+3-(2x_o^2+3)\\ &= 2(x_o^2+2hx_o+h^2)-2x_o^2\\ &= 4hx_o+2h^2.\end{aligned}$$

Thus, with any real value of x_o, the limit defined by (5.1) exists and is equal to

$$\lim_{h\to 0}\frac{4hx_o+2h^2}{h}=\lim_{h\to 0}(4x_o+2h)=4x_o.$$

This implies that the function $f(x)=2x^2+3$ is differentiable at x_o and

$$f'(x_o)=4x_o.$$

In particular, we obtain, as a special case, the result of the previous example:

$$f'(1)=4.$$ ∎

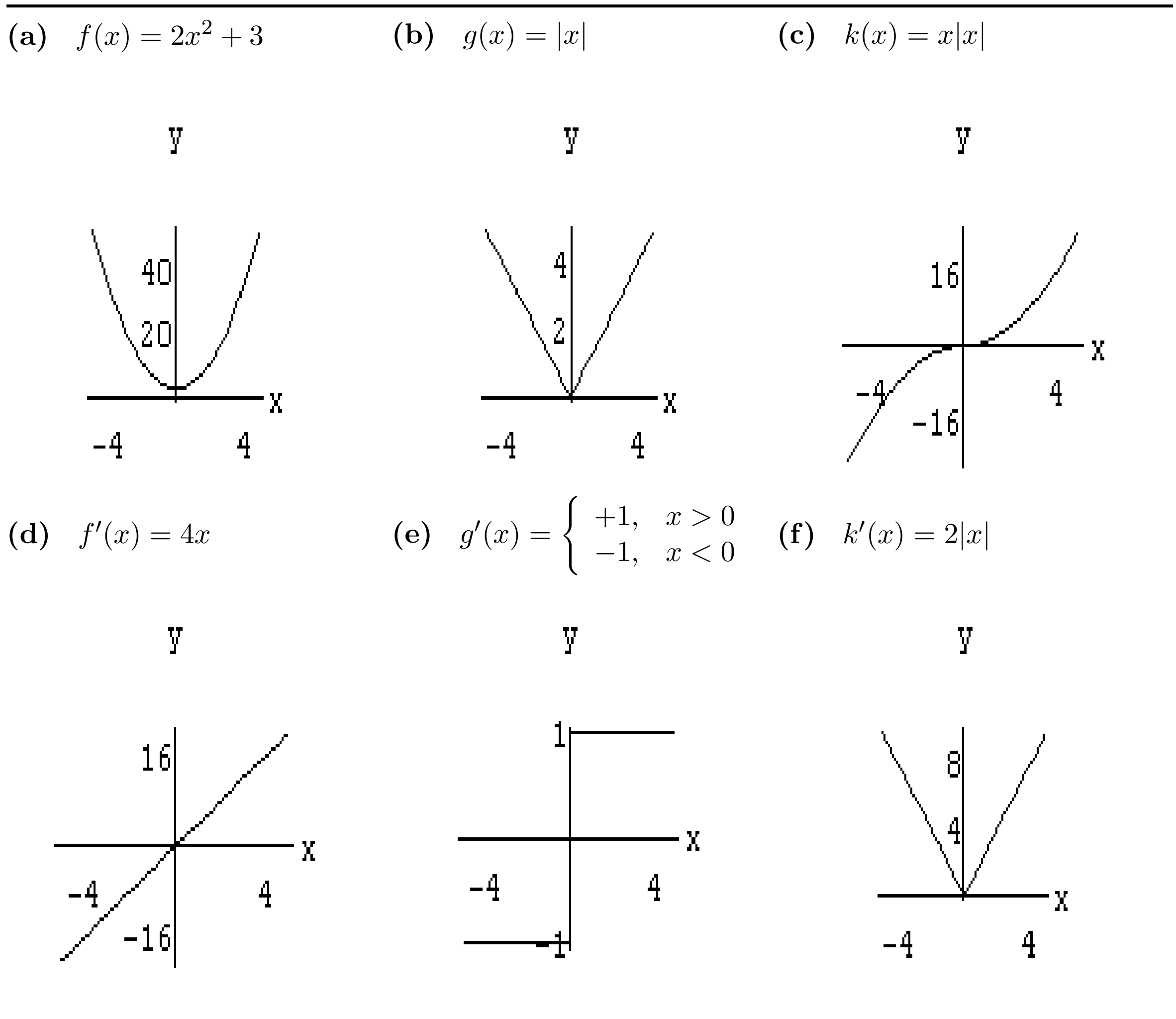


Figure 5.2: Illustrating the concept: "The derivative of a function is itself a function ."

Example 5.3 *Show that the function $g(x) = |x|$ is differentiable at any point $x = x_o \neq 0$ and it is not differentiable at the point $x = x_o = 0$.*

Solution. If $x_o = 0$ then

$$g(x_o + h) - g(x_o) = g(h) - g(0) = |h| - |0| = |h|$$

and

$$\frac{g(x_o + h) - g(x_o)}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0. \end{cases}$$

It is clear that the limit $\lim_{h \to 0} \dfrac{g(x_o + h) - g(x_o)}{h}$ does not exist, since the limit from the left is

$$\lim_{h \to 0-} \frac{g(x_o + h) - g(x_o)}{h} = -1$$

and the limit from the right is

$$\lim_{h\to 0+} \frac{g(x_o+h)-g(x_o)}{h} = 1.$$

Therefore, the function $g(x) = |x|$ does not possess a derivative at $x = 0$. Let x_o be any number different from 0. We have

$$\frac{g(x_o+h)-g(x_o)}{h} = \frac{|x_o+h|-|x_o|}{h}.$$

If $|h|$ is sufficiently small,

$$x_o > 0 \quad \Longrightarrow \quad x_o + h > 0$$

and

$$x_o < 0 \quad \Longrightarrow \quad x_o + h < 0.$$

Thus, if $x_o > 0$ then

$$g'(x_o) = \lim_{h\to 0} \frac{x_o+h-x_o}{h} = 1$$

and if $x_o < 0$ then

$$g'(x_o) = \lim_{h\to 0} \frac{-(x_o+h)-(-x_o)}{h} = -1.$$

Hence $g(x) = |x|$ is differentiable at any point $x_o \neq 0$. ∎

Example 5.4 *Show that the function $k(x) = x|x|$ is differentiable at any point $x \in \mathbb{R}$.*

Solution. For a given x_o, if $|h|$ is sufficiently small, then

$$x_o > 0 \quad \Longrightarrow \quad x_o + h > 0 \quad \text{and} \quad x_o < 0 \quad \Longrightarrow \quad x_o + h < 0.$$

We have

$$k(x_o+h)-k(x_o) = (x_o+h)|x_o+h| - x_o|x_o| = \begin{cases} (x_o+h)^2 - x_o^2 \;= 2hx_o + h^2, & \text{if } x_o > 0 \\ -(x_o+h)^2 + x_o^2 = -2hx_o - h^2, & \text{if } x_o < 0 \\ h|h|, & \text{if } x_o = 0. \end{cases}$$

Hence,

$$\lim_{h\to 0} \frac{k(x_o+h)-k(x_o)}{h} = \begin{cases} 2x_o, & \text{if } x_o > 0 \\ -2x_o, & \text{if } x_o < 0 \\ 0, & \text{if } x_o = 0. \end{cases}$$

We conclude, therefore, that the function $k(x) = x|x|$ is differentiable at any point $x = x_o$ and

$$k'(x_o) = 2|x_o|. \quad ∎$$

The derivative of the function f is itself a function; its domain is the subset of the domain of f that consists of all points x_o at which f is differentiable.

By Example 5.2 we conclude that the derivative of the function $f(x) = 2x^2 + 3$, is the function $f'(x) = 4x$. In this case the domain of f equals the domain of f'.

In Example 5.3 we conclude that the derivative of the function $g(x) = |x|$, is the function

$$g'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

In this case the domain of g is $D_g = (-\infty, \infty)$ and the domain of g' is

$$D_{g'} = D_g - \{0\} = (-\infty, 0) \cup (0, \infty).$$

Figure 5.2 shows us graphs of the functions $f(x)$, $g(x)$, and $k(x)$, and their derivatives.

Example 5.5 *Let f be a constant function, $f(x) = c$. Find the derivative f'.*

Solution. Let x_o be any real number. We have

$$f'(x_o) = \lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

Hence

$$f'(x) = (c)' = 0. \blacksquare$$

Example 5.6 *Let f be the identity function, $f(x) = x$. Find the derivative f'.*

Solution. Let x_o be any real number. We have

$$f'(x_o) = \lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} = \lim_{h \to 0} \frac{x_o + h - x_o}{h} = 1.$$

Hence,

$$f'(x) = (x)' = 1. \blacksquare$$

5.2 One-sided Derivatives

If a function f is defined on a closed interval $[a, b]$, the one-sided derivatives of f at a and b are defined by replacing the limit in (5.1) by the corresponding one-sided limit. In general, the right-hand derivative of f can be considered at any point x_o of the domain of f such that f is defined for $x_o \le x < x_o + h$ for some positive h.

Analogously, the left-hand sided derivative of f can be considered at any point x_o of the domain of f such that f is defined for $x_o - h < x \le x_o$ for some positive h.

Definition 5.2 *If $f(x)$ is defined for $x_o \leq x < x_o + h$, for some positive h, then the* **right-hand sided derivative** *of f at the point $x = x_o$ is*

$$f'_+(x_o) = \lim_{h\to 0+} \frac{f(x_o+h)-f(x_o)}{h},$$

provided that the limit exists.

If $f(x)$ is defined for $x_o - h < x \leq x_o$ for some positive h, then the **left-hand sided derivative** *of f at the point $x = x_o$ is*

$$f'_-(x_o) = \lim_{h\to 0-} \frac{f(x_o+h)-f(x_o)}{h},$$

provided that the limit exists.

Directly from the definition of the derivative of f at x_o it follows that $f'(x_o)$ exists if and only if $f'_+(x_o)$ and $f'_-(x_o)$ both exist and are equal.

Example 5.7 *Find the derivative of the function $f(x) = \sqrt{x-2}$, when $x > 2$ and investigate whether or not the right-hand sided derivative of the function at the point $x = 2$ exists.*

Solution. For any $x_o > 2$, we have

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_o+h)-f(x_o)}{h} &= \lim_{h\to 0} \frac{\sqrt{x_o-2+h}-\sqrt{x_o-2}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x_o-2+h}-\sqrt{x_o-2})(\sqrt{x_o-2+h}+\sqrt{x_o-2})}{h(\sqrt{x_o-2+h}+\sqrt{x_o-2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x_o-2+h}+\sqrt{x_o-2}} = \frac{1}{2\sqrt{x_o-2}}.\end{aligned}$$

Hence

$$f'(x) = \frac{1}{2\sqrt{x-2}}, \quad x > 2.$$

If $f(x) = \sqrt{x-2}$ and $x_o = 2$, we have

$$\frac{f(x_o+h)-f(x_o)}{h} = \frac{f(2+h)-f(2)}{h} = \frac{\sqrt{h}-\sqrt{0}}{h} = \frac{1}{\sqrt{h}}, \quad \text{provided } h > 0.$$

Thus

$$\lim_{h\to 0+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} = +\infty.$$

and we conclude that the right-hand sided derivative of $f(x) = \sqrt{x-2}$ at the point $x = 2$ does not exist. ∎

(a) $f(x) = \sqrt{x-2}$ **(b)** $f'(x) = \dfrac{1}{2\sqrt{x-2}}$

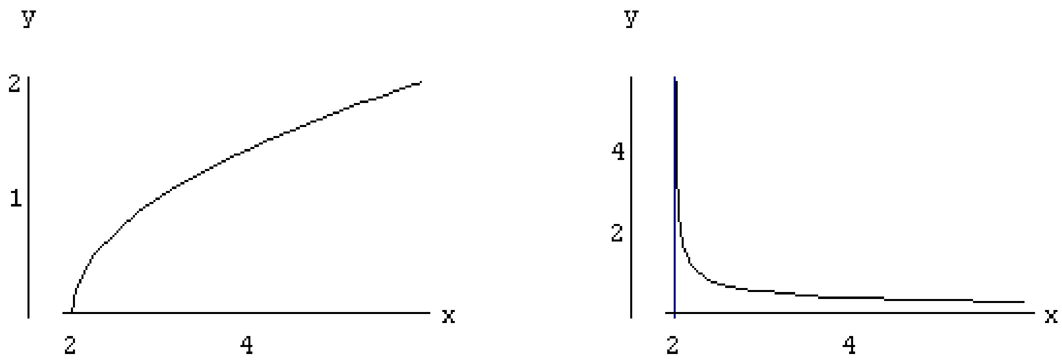


Figure 5.3: A function and its derivative.

5.3 Differentiability and Continuity

Now we will prove that any differentiable function is continuous and give some examples of functions that are continuous but yet not differentiable.

Theorem 5.2 *If f is differentiable at x_o then f is continuous at x_o.*

Proof. We have

$$f(x_o + h) = f(x_o) + Ah + \varepsilon h,$$

where $\varepsilon = \varepsilon(h) \to 0$ as $h \to 0$. Hence

$$\lim_{h\to 0} f(x_o + h) = f(x_o). \blacksquare$$

Example 5.8 *Functions that are continuous but not differentiable at the point $x = x_o$.*

(a) We shall show that the continuous function

$$f(x) = \begin{cases} x, & x \geq 2 \\ \dfrac{x^2}{2}, & x < 2 \end{cases}$$

is not differentiable at the point $x = x_o = 2$.

The one-sided derivatives of f at the point $x = x_o = 2$ are:

$$\begin{aligned} f'_+(2) &= \lim_{h\to 0+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0+} \frac{2+h-2}{h} = 1, \\ f'_-(2) &= \lim_{h\to 0-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0-} \frac{\frac{1}{2}(2+h)^2 - 2}{h} = 2. \end{aligned}$$

Since the right-hand sided and left-hand-sided derivatives of f are not equal, f does not possess a derivative at the point $x = x_o = 2$.

(b) We shall show that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous on the interval $I = (-\infty, \infty)$ but not differentiable at the point $x = x_o = 0$.

By Example 3.10, f is a continuous function on the interval $I = (-\infty, \infty)$.

Now, at the point $x = 0$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{f(h) - f(0)}{h} = \frac{h \sin \frac{1}{h}}{h} = \sin \frac{1}{h}.$$

Since $\lim_{h \to 0} \sin \dfrac{1}{h}$ does not exist, so the function f is not differentiable at the point $x = x_o = 0$.
∎

Refer to Figure 5.4(b) to see the behaviour of the function $f(x) = x \sin \frac{1}{x}$ when $x \to 0$. It is natural to accept that the tangent line to the curve $y = f(x)$ cannot de drawn. Now, in Figure 5.5, we are given another function that behaves in a similar way when $x \to 0$. We will show, however, that the function shown in Figure 5.5 possesses a derivative at the point $x = x_o = 0$.

(a) $f(x) = \begin{cases} x, & x \geq 2 \\ x^2/2, & x < 2 \end{cases}$ **(b)** $f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

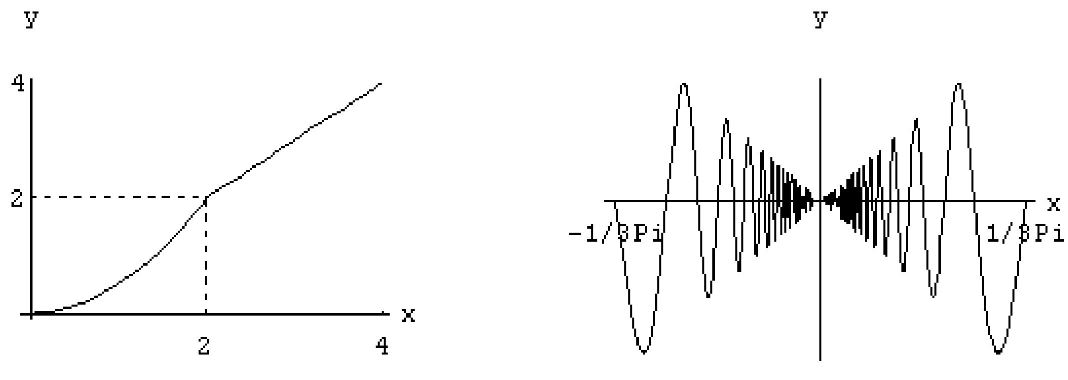


Figure 5.4: Functions that are continuous but not differentiable.

$$f(x) = \begin{cases} x^2 \sin 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

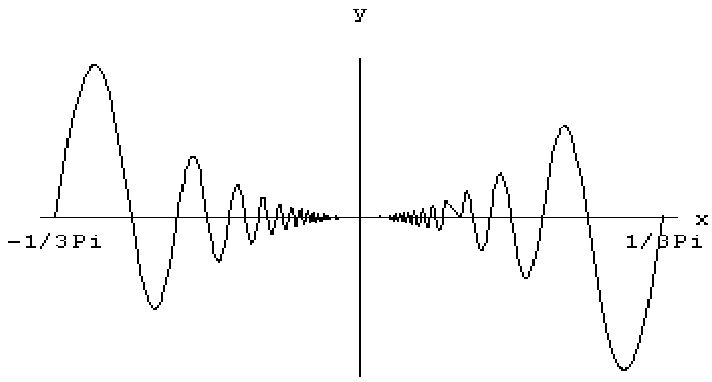


Figure 5.5: A function that is continuous and differentiable at $x = 0$.

Example 5.9 *Show that the function*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at the point $x = 0$.

Solution We have

$$\left| \frac{f(h) - f(0)}{h} \right| = \left| h \sin \frac{1}{h} \right| \leq |h| \to 0, \quad \text{as} \quad h \to 0.$$

Hence

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

The tangent line to the graph of the curve $y = f(x)$, therefore, is the horizontal axis $y = 0$. ∎

5.4 Differentiation of Elementary Functions

In this section we shall determine the derivatives of the following functions:

$$f(x) = x^n, \;\; n \in \mathbb{N}, \quad f(x) = e^x, \quad f(x) = \sin x, \quad f(x) = \cos x.$$

1. Let $f(x) = x^n$, where n is a natural number. If x is any real number then, by the binomial expansion, we have

$$(x+h)^n = x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + \binom{n}{n} h^n,$$

so that

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2} x^{n-2}h + \cdots + \binom{n}{n} h^{n-1}.$$

When $h \to 0$, all terms on the right-hand side of the above expression, except the term nx^{n-1}, converge to 0 and consequently

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Thus, if $f(x) = x^n$, where n is a positive integer, then $f(x)$ is differentiable at any point x and

$$f'(x) = (x^n)' = n\, x^{n-1}.$$

2. Let $f(x) = e^x$ and let x be any real number. Then

$$\frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h}.$$

It is shown in Example 6.8 **(h)** that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Therefore

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

Thus,

$$f'(x) = (e^x)' = e^x.$$

3. Let $f(x) = \sin x$. We have

$$\frac{\sin(x+h) - \sin x}{h} = 2\frac{\sin(h/2)\cos(x+h/2)}{h} = \frac{\sin\frac{h}{2}}{\frac{h}{2}} \cos(x + \frac{h}{2}).$$

When $h \to 0$, then $\frac{h}{2} \to 0$ and, by Example 3.7,

$$\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1.$$

Making use of the fact that $\cos x$ is continuous, we obtain

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \lim_{h\to 0} \cos(x + \frac{h}{2}) = 1 \cdot \cos x = \cos x.$$

Alternatively,

$$\begin{aligned}\frac{\sin(x+h)-\sin x}{h} &= \frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\ &= \frac{\sin h}{h}\cos x-\sin x\frac{1-\cos h}{h}\\ &= \frac{\sin h}{h}\cos x-\sin x\frac{2\sin^2 h/2}{h}\\ &= \frac{\sin h}{h}\cos x-\frac{h}{2}\sin x\left(\frac{\sin h/2}{h/2}\right)^2\longrightarrow \cos x,\quad \text{as}\quad h\longrightarrow 0.\end{aligned}$$

Hence

$$f'(x)=(\sin x)'=\cos x.$$

4. Let $f(x)=\cos x$. We have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{2}{h}\sin\frac{2x+h}{2}\sin(-\frac{h}{2})\\ &= -\lim_{h\to 0}\sin\frac{2x+h}{2}\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\\ &= -\sin x.\end{aligned}$$

Note that $\lim_{h\to 0}\sin\frac{2x+h}{2}=\sin x$, for $\sin x$ is continuous over the interval $(-\infty,\infty)$. Therefore,

$$f'(x)=(\cos x)'=-\sin x.$$

5.5 The Algebra of Derivatives

Theorem 5.3 The rules for differentiation.
Suppose that the functions f and g defined on a given interval I are differentiable at some point $x_o\in I$. Then:

(a) *Addition Rule: The function $f+g$ is differentiable at x_o, and*

$$(f+g)'(x_o)=f'(x_o)+g'(x_o);$$

(b) *Product Rule: The function $f\cdot g$ is differentiable at x_o, and*

$$(f\cdot g)'(x_o)=f(x_o)g'(x_o)+f'(x_o)g(x_o);$$

(c) *Scalar Product Rule: The function $h(x)=cf(x)$, where c is a constant, is differentiable at x_o, and*

$$h'(x_o)=(c\,f)'(x_o)=c\,f'(x_o);$$

(d) *Reciprocal Rule:* *If $g(x_o) \neq 0$, then $\frac{1}{g}$ is differentiable at x_o, and*

$$\left(\frac{1}{g}\right)'(x_o) = -\frac{g'(x_o)}{[g(x_o)]^2};$$

(e) *Quotient Rule:* *If $g(x_o) \neq 0$, then $\frac{f}{g}$ is differentiable at x_o, and*

$$\left(\frac{f}{g}\right)'(x_o) = \frac{f'(x_o)g(x_o) - f(x_o)g'(x_o)}{[g(x_o)]^2}.$$

Proof. By assumption, $f'(x_o) = A$ and $g'(x_o) = B$ exist, so that, by Theorem 5.1, we have

$$\begin{aligned} f(x_o + h) &= f(x_o) + A\,h + h\,\varepsilon_1(h), \\ g(x_o + h) &= g(x_o) + B\,h + h\,\varepsilon_2(h), \end{aligned}$$

where $\lim_{h\to 0} \varepsilon_1(h) = 0$, $\lim_{h\to 0} \varepsilon_2(h) = 0$.

(a)
We have

$$(f + g)(x_o + h) = (f + g)(x_o) + (A + B)h + h(\varepsilon_1(h) + \varepsilon_2(h)).$$

Hence $(f+g)'(x_o)$ exists and equals $A+B$, since $\varepsilon_3 = \varepsilon_1(h)+\varepsilon_2(h)$ tends to zero as h tends to zero.

(b) We have

$$\begin{aligned} f \cdot g(x_o + h) &= (f(x_o) + Ah + h\varepsilon_1(h))(g(x_o) + Bh + h\varepsilon_2(h)) \\ &= f(x_o)g(x_o) + (Ag(x_o) + Bf(x_o))h + h\varepsilon_3(h), \end{aligned}$$

where $\varepsilon_3(h) = A\varepsilon_2(h) + B\varepsilon_1(h) + h\varepsilon_1(h)\varepsilon_2(h)$.

Hence $(f \cdot g)'(x_o) = Ag(x_o) + Bf(x_o)$, since $\lim_{h\to 0} \varepsilon_3(h) = 0$.

(c) Let $g(x) = c$, so that

$$h(x) = c\,f(x) = f(x)g(x), \quad -\infty < x < \infty.$$

Then,

$$g'(x_o) = 0,$$

and an application of Product Rule **(b)** gives

$$h'(x_o) = f(x_o)g'(x_o) + f'(x_o)g(x_o) = 0 \cdot f(x_o) + cf'(x_o) = c\,f'(x_o).$$

(d) First, we show that the function $1/g$ is well defined for $x = x_o + h$, when h is sufficiently small. We argue as follows. Since g is differentiable at x_o, by Theorem 5.2, g is continuous at x_o. By Lemma 4.1, therefore, there exists $\delta > 0$ such that $g(x_o + h) \neq 0$ for $|x_o + h - x_o| = |h| < \delta$. Hence the expression

$$\frac{\left(\frac{1}{g}\right)(x_o + h) - \left(\frac{1}{g}\right)(x_o)}{h}$$

is defined for sufficiently small h. We have

$$\begin{aligned} \left(\frac{1}{g}\right)(x_o + h) - \left(\frac{1}{g}\right)(x_o) &= -\frac{g(x_o + h) - g(x_o)}{g(x_o + h)g(x_o)} \\ &= -\frac{Ah + h\varepsilon_1(h)}{(g(x_o) + Ah + h\varepsilon_1(h))g(x_o)} \\ &= -\frac{A}{(g(x_o))^2}h + h\varepsilon(h), \end{aligned}$$

where

$$\varepsilon(h) = \frac{Ah + h\varepsilon_1(h) - \varepsilon_1(h)g(x_o)}{g(x_o)(g(x_o) + Ah + h\varepsilon_1(h))} \to 0, \quad \text{as} \quad h \to 0.$$

Hence

$$\left(\frac{1}{g}\right)'(x_o) = -\frac{g'(x_o)}{g(x_o)^2}.$$

(e) Applying Product Rule **(b)** and Reciprocal Rule **(d)**, we get

$$\begin{aligned} \left(\frac{f}{g}\right)'(x_o) &= \left(f \cdot \frac{1}{g}\right)'(x_o) \\ &= f'(x_o) \cdot \left(\frac{1}{g}\right)(x_o) + f(x_o) \cdot \left(\frac{1}{g}\right)'(x_o) \\ &= \frac{f'(x_o)}{g(x_o)} - f(x_o)\frac{g'(x_o)}{[g(x_o)]^2} \\ &= \frac{f'(x_o)g(x_o) - f(x_o)g'(x_o)}{[g(x_o)]^2}. \quad \blacksquare \end{aligned}$$

Application of Theorem 5.3 allows us to differentiate easily some functions for which it would be difficult to obtain the limit

$$\lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h}.$$

Example 5.10 *Using rules for differentiation .*

(a) If $f(x) = 2x^3 + 7x^2 + 5$, then

$$f'(x) = 2(x^3)' + 7(x^2)' + (5)' = 2(3x^2) + 7(2x) + 0 = 6x^2 + 14x.$$

(b) If $g(x) = x^{-n}, \quad n \in \mathbb{N}$, then

$$g'(x) = (x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

(c) If $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$, then

$$h'(x) = \frac{(x^2-1)'(x^2+1) - (x^2-1)(x^2+1)'}{(x^2+1)^2} = \frac{2x(x^2+1) - (x^2-1) \cdot 2x}{(x^2+1)^2}$$
$$= \frac{4x}{(x^2+1)^2}.$$

(d) If $u(x) = \tan x$, then provided that $\cos x \neq 0$, i.e. $x \neq \frac{\pi}{2} + k\pi, \quad k = \pm 1, \pm 2, \ldots,$

$$u'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

(e) If $v(x) = \cot x$, then provided that $\sin x \neq 0$ or $x \neq k\pi, \quad k = \pm 1, \pm 2, \ldots,$

$$v'(x) = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}. \quad \blacksquare$$

5.6 Differentiation of Composite Functions

Theorem 5.4 The Chain Rule
Consider a function g with domain $I \subset \mathbb{R}$ and range J and a function f defined on the interval J. If g is differentiable at the point $x_o \in I$ and f is differentiable at the point $y_o = g(x_o) \in J$, then the composite function $f \circ g$ is differentiable at the point x_o and

$$(f \circ g)'(x_o) = f'(y_o) g'(x_o).$$

Proof. Let $A = f'(y_o)$ and let $B = g'(x_o)$. By Theorem 5.1 we have

$$f(y_o + k) = f(y_o) + Ak + \varepsilon_1 k, \qquad g(x_o + h) = g(x_o) + Bh + \varepsilon_2 h,$$

where

$$\lim_{k \to 0} \varepsilon_1 = \lim_{k \to 0} \varepsilon_1(k) = 0, \qquad \lim_{h \to 0} \varepsilon_2 = \lim_{h \to 0} \varepsilon_2(h) = 0.$$

Now

$$(f \circ g)(x_o + h) = f(g(x_o + h)) = f[g(x_o) + (B + \varepsilon_2)h] = f(y_o + k),$$

where $k = h(B + \varepsilon_2)$.

We have

$$f(y_o + k) = f(y_o) + k(A + \varepsilon_1) = f(y_o) + (A + \varepsilon_1)(B + \varepsilon_2)h = f(y_o) + ABh + \varepsilon_3 h.$$

Therefore

$$(f \circ g)(x_o + h) = f(g(x_o)) + ABh + \varepsilon_3 h,$$

where $\varepsilon_3 = A\varepsilon_2 + B\varepsilon_1 + \varepsilon_1\varepsilon_2 \to 0$, as $h \to 0$. Hence, applying Theorem 5.1 again, we conclude that $(f \circ g)'(x_o) = AB$, as required. ∎

Example 5.11 *Find derivative of the function* $H(x) = e^{1-x^3}$.

Solution. H can be considered as a composite function $f \circ g$, where

$$f(y) = e^y, \quad -\infty < y < \infty,$$
$$y = g(x) = 1 - x^3, \quad -\infty < x < \infty.$$

We have

$$f'(y) = e^y,$$
$$g'(x) = -3x^2.$$

Hence

$$H'(x) = (f \circ g)'(x) = f'(y)g'(x) = e^y \cdot (-3x^2) = -3x^2 e^{1-x^3}, \quad -\infty < x < \infty.$$ ∎

(a) $H(x) = e^{1-x^3}$ **(b)** $H'(x) = -3x^2 e^{1-x^3}$

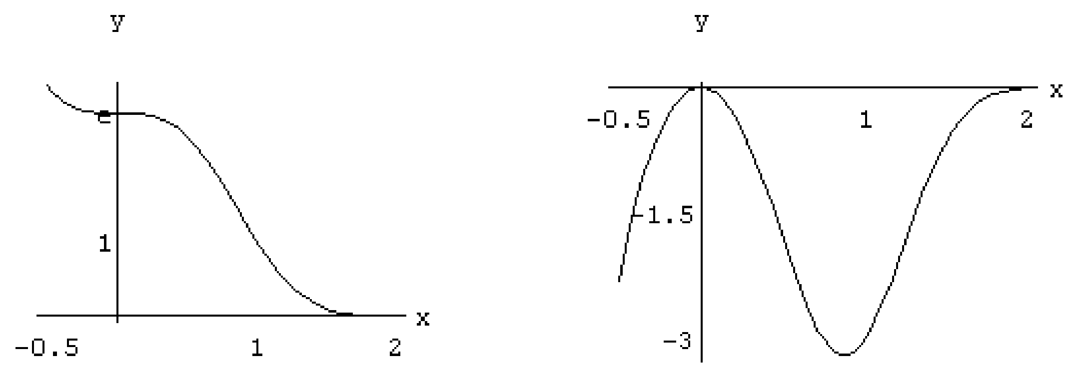


Figure 5.6: Derivative of a composite function.

Example 5.12 *Determine derivative of the function* $H(x) = \sin(\cos x)$ *at a given point* $x = x_o$.

Solution. Let $y_o = g(x_o) = \cos x_o$. Then we have

$$H'(x_o) = (f \circ g)'(x_o) = f'(y_o)g'(x_o) = \cos y_o \cdot (-\sin x_o) = -\sin x_o \cdot \cos(\cos x_o).$$ ∎

(a) $H(x) = \sin(\cos x)$ **(b)** $H'(x) = -\sin x \ \cos(\cos x)$

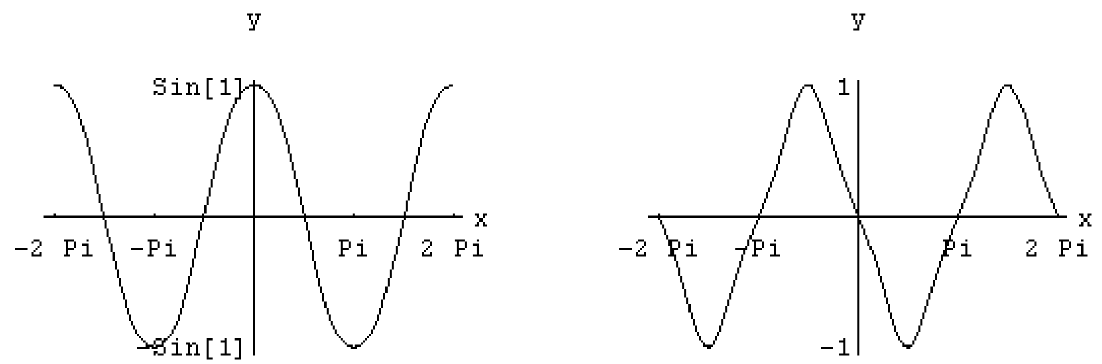


Figure 5.7: Derivative of a composite function.

5.7 Differentiation of Inverse Functions

Theorem 5.5 *Suppose that the function $y = f(x)$ with domain $I \subset \mathbb{R}$ and range $J \subset \mathbb{R}$, is strictly monotone and let $x = g(y)$ be the inverse function to f.*

If f is differentiable at a given point $x_o \in I$ and $f'(x_o) \neq 0$, then g is differentiable at the point $y_o = f(x_o)$ and

$$g'(y_o) = \frac{1}{f'(x_o)} = \frac{1}{f'(g(y_o))}.$$

Proof. Let $y_o = f(x_o)$. Then $g(y_o) = x_o$. Given $k \neq 0$, let

$$h = g(y_o + k) - g(y_o). \tag{5.2}$$

We have

$$f(x_o + h) = f(x_o + g(y_o + k) - g(y_o)) = f(g(y_o + k)) = y_o + k.$$

Since $f'(x_o)$ exists, we can write

$$f(x_o + h) = f(x_o) + Ah + h\varepsilon_1(h), \text{ where } A = f'(x_o) \neq 0, \ \varepsilon_1(h) \to 0 \text{ as } h \to 0.$$

Hence

$$y_o + k = f(x_o) + Ah + h\varepsilon_1(h),$$

which implies that

$$k = Ah + h\varepsilon_1(h),$$

or

$$h = \frac{k}{A + \varepsilon_1(h)}.$$

Therefore, using (5.2), we get

$$g(y_o + k) = g(y_o) + h = g(y_o) + \frac{1}{A}k + k\varepsilon(k), \tag{5.3}$$

where

$$\varepsilon(k) = \frac{h - k/A}{k} = \frac{1}{A + \varepsilon_1(h)} - \frac{1}{a} = \frac{1}{A + \varepsilon_1(h)} \frac{\varepsilon_1(h)}{A} \to 0, \quad \text{as} \quad k \to 0.$$

Clearly, (5.3) implies that g is differentiable at the point y_o and

$$g'(y_o) = \frac{1}{A},$$

which completes the proof. ∎

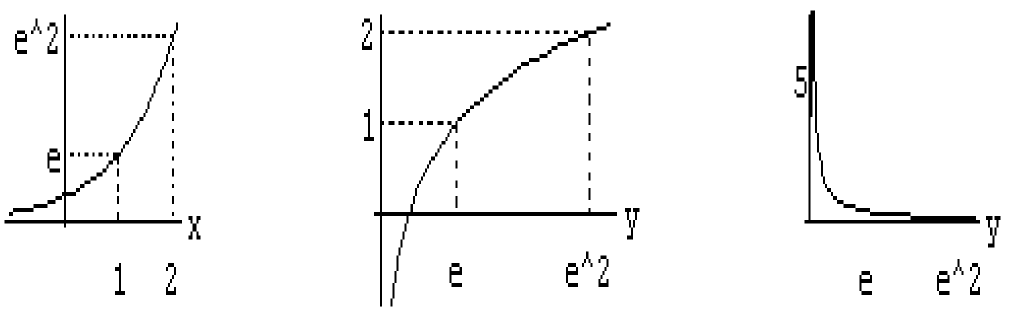
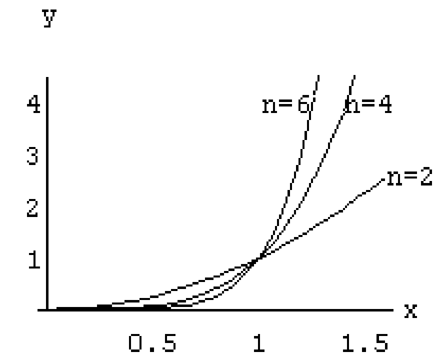
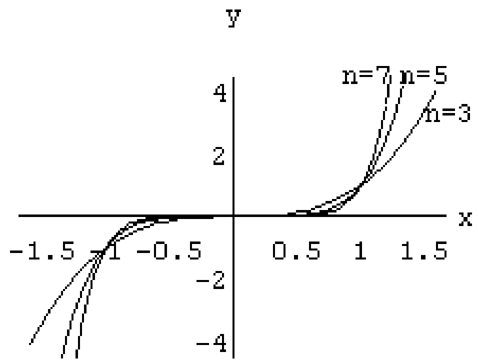


Figure 5.8: The exponential function $f(x) = e^x$, its inverse $g(y)$, and the derivative of the inverse.

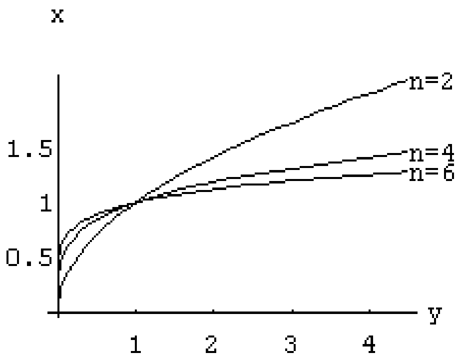
(i) $f(x) = x^n$, n even;



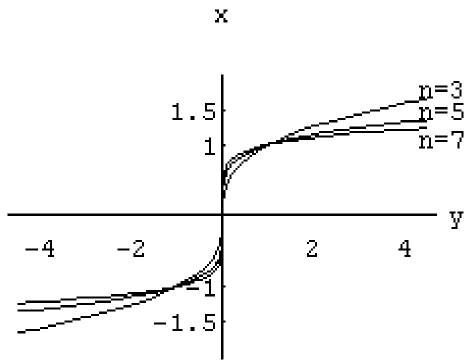
(ii) $f(x) = x^n$, n odd;



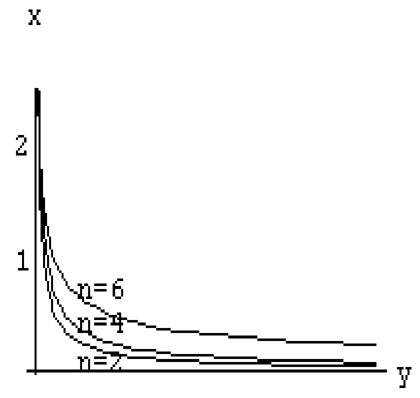
(iii) $g(y) = \sqrt[n]{y}$, n even;



(iv) $g(y) = \sqrt[n]{y}$, n odd;



(v) $g'(y) = \dfrac{1}{n} y^{1/n-1}$, n even;



(vi) $g'(y) = \dfrac{1}{n} y^{1/n-1}$, n odd.

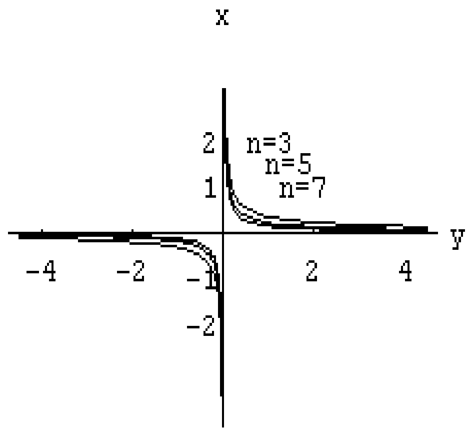
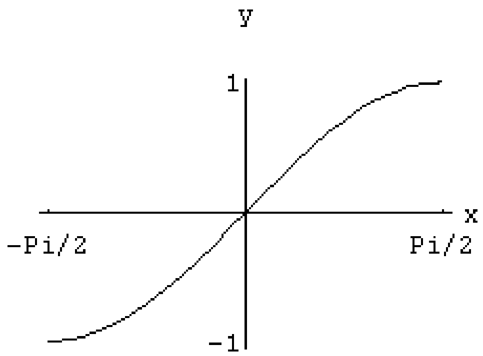
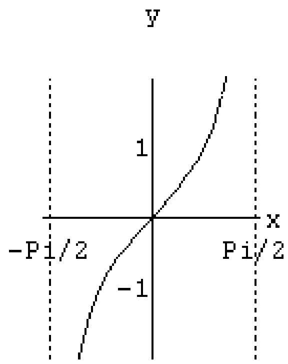


Figure 5.9: The power function $f(x) = x^n$, its inverse $g(y)$, and the derivative of the inverse.

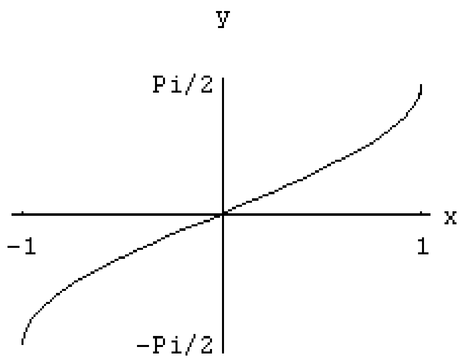
(i) $f(x) = \sin x$;



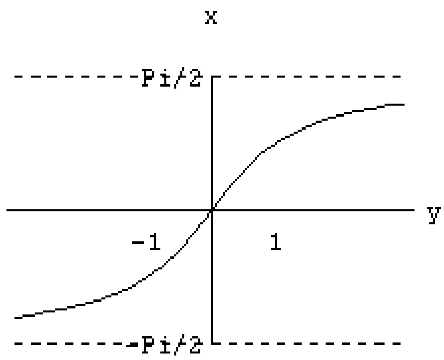
(ii) $f(x) = \tan x$;



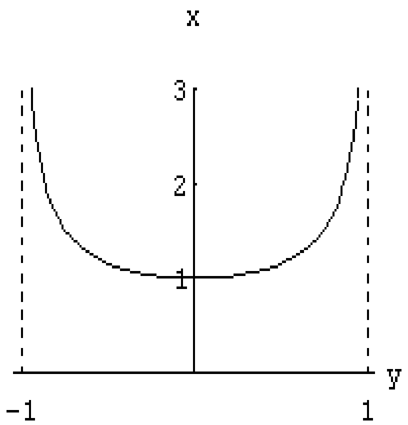
(iii) $g(y) = \arcsin y$;



(iv) $g(y) = \arctan y$;



(v) $g'(y) = \dfrac{1}{\sqrt{1-y^2}}$;



(vi) $g'(y) = \dfrac{1}{1+y^2}$.

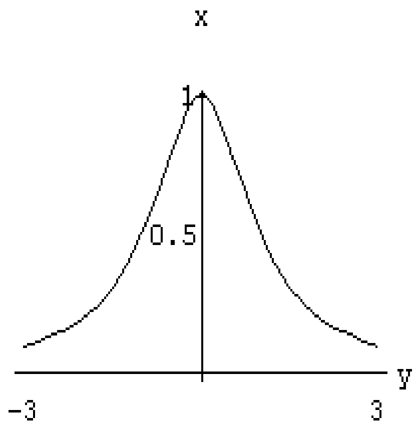


Figure 5.10: Trigonometric functions: sine and tangent; their inverses and the derivatives of the inverse functions.

Example 5.13 *Finding derivatives of inverse functions.*

(a) Consider the function $y = f(x) = e^x$. f is one-to-one on the interval $I = (-\infty, \infty)$ and maps I onto $J = (0, \infty)$. We have

$$f'(x) = (e^x)' = e^x, \quad x \in I,$$

and we can see that $f'(x) \neq 0, \quad x \in I$. The inverse function $g(y) = \log_e y = \log y$, therefore, is differentiable on J and we obtain

$$g'(y) = (\log y)' = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{y}.$$

Figure 5.8 shows us the function $f(x) = e^x$, for $-2 < x < 2$, its inverse $g(y) = \log y$, and the derivative $g'(x)$.

(b) Let $f(x) = x^n$, where n is even. Let $I = [0, \infty]$. Clearly f is one-to-one on the interval I and maps it onto $J = [0, \infty)$. The inverse function is $g(y) = y^{1/n} = \sqrt[n]{y}, \quad y \in J$. We have $f'(x) = nx^{n-1} \neq 0$ for all $x \in I, x \neq 0$. Hence $g(y)$ is differentiable over the interval $(0, \infty)$ and its derivative is

$$g'(y) = (y^{1/n})' = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{ny^{(n-1)/n}} = \frac{1}{n} y^{\frac{1}{n}-1}, \quad y > 0.$$

Figure 5.9(i), (iii), and (v) shows us graphs of the function $f(x) = x^n$, when n is even, along with the inverse function $g(y) = \sqrt[n]{y}$ and the derivative $g'(y)$.

(c) Let $f(x) = x^n$, where n is odd, $n \neq 1$. Then $f(x)$ is one-to-one on the whole real line, $I = (-\infty, \infty)$. We can see that $f'(x) = nx^{n-1} \neq 0$ for all $x \in (-\infty, \infty)$, except $x = 0$. Now f maps I onto $J = (-\infty, \infty)$, and the inverse function $g(y) = y^{1/n} = \sqrt[n]{y}, \quad y \in J$, is differentiable for $y \in J, \quad y \neq 0$. We have

$$g'(y) = (y^{1/n})' = \frac{1}{n} y^{\frac{1}{n}-1}, \quad y \neq 0,$$

provided n is odd. We note that $g'(0)$ does not exist. See Figure 5.9(ii), (iv), and (vi).

Now refer to Figure 5.10 which gives us graphs of the trigonometric functions $\sin x$ and $\tan x$, along with their inverses and the derivatives of the inverse functions.

(d) Consider the function $y = f(x) = \sin x$ on the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. f is strictly increasing on I and maps I onto $J = [-1, 1]$. The inverse function is

$$f^{-1}(y) = g(y) = \arcsin y, \quad y \in J.$$

We have $f'(x) = \cos x \neq 0$ for $x \in I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence the inverse function g is differentiable at any point $y \in (-1, 1)$ and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}, \quad -1 < y < 1.$$

(e) If $y = f(x) = \cos x$, $x \in I = [0, \pi]$, then $f^{-1}(y) = g(y) = \arccos y$, $y \in J = [-1, 1]$. Since $f'(x) = (\cos x)' = -\sin x \neq 0$ for $0 < x < \pi$, we get

$$g'(y) = (\arccos y)' = \frac{1}{f'(x)} = -\frac{1}{\sin x} = -\frac{1}{\sqrt{1-y^2}}, \quad -1 < y < 1.$$

(f) Similarly we differentiate the inverse function of $f(x) = \tan x$ on the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$g'(y) = (\arctan y)' = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

(g) Similarly we obtain

$$(\text{arccot } y)' = -\frac{1}{1+y^2}, \quad -\infty < x < \infty. \quad \blacksquare$$

5.8 Derivatives of Higher Order

If the derivative f' of a given differentiable function f is itself differentiable, it is said that f is twice differentiable and the derivative of f' is called the second derivative of f and is denoted by f''. The domain of f'' is the set I, $I \subset \mathbb{R}$, where the derivative of the function f' exists. If the derivative of the second derivative of f exists, it is said that f is three times differentiable and the derivative of f'' is called the third derivative of f and is denoted by f''' or $f^{(3)}$.

The process of differentiation may continue to yield the 4th, 5th, ..., nth derivatives of f, normally denoted by $f^{(4)}, f^{(5)}, \ldots, f^{(n)}$.

Other notations for the nth derivative ($n = 2, 3, \ldots$) of the function $y = f(x)$ are

$$\frac{d^n}{dx^n} f(x), \quad \frac{d^n f(x)}{dx^n}, \quad y^{(n)}, \quad \frac{d^n y}{dx^n}.$$

The various functions $f^{(n)}, n \geq 2$, are called higher-order derivatives of f. By convention, we shall write $f^{(0)}$ and $f^{(1)}$ to denote f and f', respectively.

Example 5.14 *Find all higher-order derivatives of the function*

$$f(x) = 3x^4 + 2x - 1.$$

Solution. For any value of x we have

$$\begin{aligned}
f^{(1)}(x) &= (3x^4 + 2x - 1)' = 12x^3 + 2, \\
f^{(2)}(x) &= (12x^3 + 2)' = 36x^2, \\
f^{(3)} &= (36x^2)' = 72x, \\
f^{(4)} &= (72x)' = 72, \\
f^{(n)} &= 0, \quad n \geq 5.
\end{aligned}$$

$\blacksquare$

Example 5.15 *Find the first and second derivatives of the function*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Solution. By Example 5.9, $f'(0) = 0$.
If $x \neq 0$ then

$$\begin{aligned} f'(x) &= (x^2 \sin \frac{1}{x})' = 2x \sin \frac{1}{x} + x^2 (\sin \frac{1}{x})' = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} (-\frac{1}{x^2}) \\ &= 2x \sin \frac{1}{x} - \cos \frac{1}{x}. \end{aligned}$$

Hence

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

If $x \neq 0$ then

$$\begin{aligned} f''(x) &= \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)' \\ &= 2 \sin \frac{1}{x} + 2x \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) + \sin x \left(-\frac{1}{x^2}\right) \\ &= 2 \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} - \frac{1}{x^2} \sin \frac{1}{x}. \end{aligned}$$

Now,

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h},$$

provided that the limit exist. We have

$$\frac{f'(h) - f'(0)}{h} = 2 \sin \frac{1}{h} - \frac{1}{h} \cos \frac{1}{h}.$$

Since

$$\left| \frac{f'(h) - f'(0)}{h} \right| = \left| \frac{1}{h} \cos \frac{1}{h} - 2 \sin \frac{1}{h} \right| \geq \frac{1}{|h|} \left| \cos \frac{1}{h} \right| - 2 \left| \sin \frac{1}{h} \right| \geq \frac{1}{|h|} - 2,$$

for all sufficiently small h, hence $f''(0)$ does not exist.

In fact, the range of values of $\frac{f'(h)-f'(0)}{h}$, as h approaches 0 is $(-\infty, \infty)$. ∎

(a) $f(x)$ **(b)** $f'(x)$ **(c)** $f''(x)$

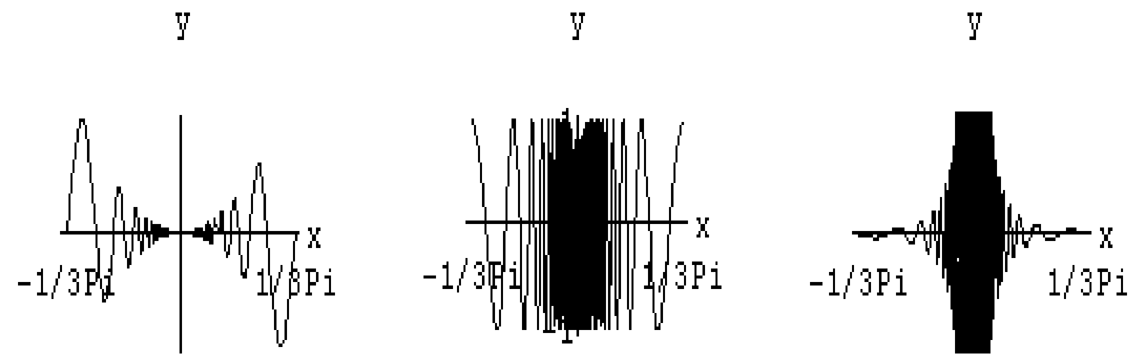


Figure 5.11: The function $f(x)$ of Example **5.15** and its first and second derivatives.

Example 5.16

If $f(x) = \sin x$, then we have

$$f^{(1)}(x) = \cos x = \sin(x + \pi/2),$$

$$f^{(2)}(x) = -\sin(x) = \sin(x + 2\pi/2),$$

$$f^{(3)}(x) = -\cos x = \sin(x + 3\pi/2)\ .$$

Therefore

$$f^{(n)}(x) = \sin(x + n\pi/2), \quad n = 0, 1, 2, \ldots \quad \blacksquare$$

Example 5.17

If $f(x) = \cos x$, then we have

$$f^{(1)}(x) = -\sin x = \cos(x + \pi/2),$$

$$f^{(2)}(x) = -\cos(x) = \cos(x + 2\pi/2),$$

$$f^{(3)}(x) = \sin x = \cos(x + 3\pi/2)\ldots.$$

Therefore

$$f^{(n)}(x) = \cos(x + n\pi/2), \quad n = 0, 1, 2, \ldots \quad \blacksquare$$

Example 5.18

Let $f(x) = \log(1+x)$. We have

$$f^{(1)}(x) = (1+x)^{-1},$$
$$f^{(2)}(x) = -1 \cdot (1+x)^{-2},$$
$$f^{(3)}(x) = 1 \cdot 2 \cdot (1+x)^{-3},$$
$$f^{(4)}(x) = -1 \cdot 2 \cdot 3 \cdot (1+x)^{-4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}, \quad n = 1, 2, \ldots \quad \blacksquare$$

Example 5.19

Let $f(x) = (1+x)^{\alpha}$, where α is any real number. Then

$$f^{(1)}(x) = \alpha(1+x)^{\alpha-1}$$
$$f^{(2)}(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$
$$f^{(3)}(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$
$$f^{(4)}(x) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}. \quad \blacksquare$$

5.9 Leibniz's Formula

Now we shall prove a useful formula for the n-th derivative of the product $f \cdot g$ of two functions that are n times differentiable.

Theorem 5.6 Leibniz's Formula *If $f^{(n)}(a)$ and $g^{(n)}(a)$ exist, then the product $f \cdot g$ is n times differentiable and*

$$(f \cdot g)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a). \tag{5.4}$$

Proof. (By mathematical induction)
Step 1: If $n = 1$ then (5.4) reduces to the product rule of differentiation:

$$(f \cdot g)'(a) = f(a)g'(a) + f'(a)g(a)$$

that has already been proved (see page 106).

Step 2: We need to prove the implication

$$T(n) \implies T(n+1),$$

where $T(n)$ stands for (5.4), so that

$$T(n+1) \equiv (f \cdot g)^{(n+1)}(a) = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a) g^{(n+1-k)}(a).$$

Making use of the assumption that (5.4) is true and using the product rule of differentiation, we obtain

$$\begin{aligned}
(f \cdot g)^{(n+1)}(a) &= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)\, g^{(n-k)}(a) \right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(a)\, g^{(n-k)}(a) \right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(a)\, g^{(n+1-k)}(a) + f^{(k+1)}(a)\, g^{(n-k)}(a) \right)
\end{aligned}$$

To simplify notations, we will use $f^{(i)}$ and $g^{(i)}$ instead of $f^{(i)}(a)$ and $g^{(i)}(a)$, $i = 1, 2, \ldots, n+1$, respectively. Hence,

$$\begin{aligned}
(f \cdot g)^{(n+1)}(a) &= \binom{n}{0} f\, g^{(n+1)} + \binom{n}{0} f'\, g^{(n)} \\
&+ \binom{n}{1} f'\, g^{(n)} + \binom{n}{1} f''\, g^{(n-1)} \\
&+ \binom{n}{2} f''\, g^{(n-1)} + \binom{n}{2} f^{(3)}\, g^{(n-2)} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \binom{n}{n-1} f^{(n-1)}\, g'' + \binom{n}{n-1} f^{(n)}\, g' \\
&+ \binom{n}{n} f^{(n)}\, g' + \binom{n}{n} f^{(n+1)}\, g.
\end{aligned}$$

Recall the following identities

$$\begin{aligned}
\binom{n}{0} &= \binom{n+1}{0} = 1 \\
\binom{n}{n} &= \binom{n+1}{n+1} = 1 \\
\binom{n}{k} + \binom{n}{k+1} &= \binom{n+1}{k+1}, \quad k = 0, 1, 2, \ldots, n-1,
\end{aligned}$$

to conclude that

$$\begin{aligned}(f\cdot g)^{(n+1)}(a) &= \binom{n+1}{0} f^{(0)}g^{(n+1)} + \binom{n+1}{1} f^{(1)}g^{(n)} + \cdots + \binom{n+1}{n+1} f^{(n+1)}g^{(0)} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a)g^{(n+1-k)}(a). \quad \blacksquare\end{aligned}$$

Example 5.20 *Using Leibniz's formula.*

Find the 4-th derivative of the function

$$H(x) = x^3 e^x.$$

Solution.
Let $f(x) = x^3$ and $g(x) = e^x$, so that $H(x) = f(x) \cdot g(x)$. We have

$$\begin{aligned} f^{(0)}(x) &= x^3, \\ f^{(1)}(x) &= 3x^2, \\ f^{(2)}(x) &= 6x, \\ f^{(3)}(x) &= 6, \\ f^{(4)}(x) &= 0, \\ \\ g^{(k)}(x) &= e^x, \quad k = 0, 1, 2, 3, 4. \end{aligned}$$

Hence

$$\begin{aligned} H^{(4)}(x) &= (f\cdot g)^{(4)}(x) \\ &= f^{(0)}(x)g^{(4)}(x) + 4f^{(1)}(x)g^{(3)}(x) + 6f^{(2)}(x)g^{(2)}(x) \\ &\quad + 4f^{(3)}(x)g^{(1)}(x) + f^{(4)}(x)g^{(0)}(x) \\ &= e^x\left(x^3 + 12x^2 + 36x + 24\right). \end{aligned}$$

Therefore $H^{(4)}(x) = \left(x^3 e^x\right)^{(4)} = e^x\left(x^3 + 12x^2 + 36x + 24\right)$. $\blacksquare$

5.10 Extreme Values

In Section 3.3, we defined the minimum and maximum of a function f on a given set S. If f attains its minimum on S at the point x_1, then

$$\forall x \in S \;\; f(x_1) \le f(x); \tag{5.5}$$

if f attains its maximum on S at the point x_2, then

$$\forall x \in S \;\; f(x_2) \ge f(x). \tag{5.6}$$

If the inequality (5.5) is satisfied for all x in a neighbourhood of a given point x_o (that is contained in S), then we say that f has a **local minimum** at the point x_o. If the inequality (5.6) is satisfied for all x in a neighbourhood of the point x_o (that is contained in S), then we say that f has a **local maximum** at x_o.

Definition 5.3 *Let f be a function defined on a set S. A point $x_o \in S$ is a* **local minimum/maximum point** *for the function f on S if there is some $\delta > 0$ such that x_o is a minimum/maximum point for f on the set $S \cap (x_o - \delta, x_o + \delta)$.*

The value $f(x_o)$ is then called a **local maximum/local minimum** *or, simply, an* **extreme value** *of the function f.*

Theorem 5.7 Local Extremum Theorem *If f has an extreme value at the point x_o and $f'(x_o)$ exists, then*

$$f'(x_o) = 0.$$

Proof. Suppose that f has a local maximum at x_o so that there exists $\delta > 0$ such that if $0 < h < \delta$ then

$$f(x_o + h) - f(x_o) \geq 0,$$
$$f(x_o - h) - f(x_o) \geq 0.$$

Dividing the above inequalities by h and by $k = -h$, respectively, gives

$$\frac{f(x_o + h) - f(x_o)}{h} \geq 0 \quad \text{and} \quad \frac{f(x_o + k) - f(x_o)}{k} \leq 0,$$

for $0 < h < \delta$ and $-\delta < k < 0$. Since $f'(x_o)$ exists, the one-sided derivatives of f at x_o exist and are equal, we arrive at the conclusion

$$f'(x_o) = f'_+(x_o) = \lim_{h \to 0+} \frac{f(x_o + h) - f(x_o)}{h} \geq 0$$
$$f'(x_o) = f'_-(x_o) = \lim_{k \to 0-} \frac{f(x_o + k) - f(x_o)}{k} \leq 0.$$

Hence, $f'(x_o) = 0$.

Analogously, if $f(x)$ has a local minimum at x_o, the same conclusion holds. ∎

5.11 Rolle's Theorem

Theorem 5.8 Rolle's Theorem *Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there exists an x_o, $a < x_o < b$, such that $f'(x_o) = 0$.*

Proof. Since f is continuous on a closed interval, by the boundedness theorem, it attains a minimum value and a maximum value on this interval. Let

$$m = \min_{x \in [a,b]} f(x) = f(x_1), \quad M = \max_{x \in [a,b]} f(x) = f(x_2).$$

We can assume that $m \neq M$, since if this is not the case then f is a constant function, $f(x) = c, \;\; a \leq x \leq b$, which implies that $f'(x) = 0, \;\; a \leq x \leq b$, and then the theorem is obviously true.

Since $f(x_1) = m \neq M = f(x_2)$ and $f(a) = f(b)$, it follows that at least one of the points x_1 or x_2 is not the end-point of the interval $[a, b]$.

If $x_1 \in (a,b)$, then by the hypothesis, $f'(x_1)$ exists, and by the local extremum theorem we conclude that $f'(x_1) = 0$.

If $x_2 \in (a,b)$, then by the hypothesis, $f'(x_2)$ exists, and by the local extremum theorem we conclude that $f'(x_2) = 0$. ∎

Theorem 5.9 The Cauchy Mean Value Theorem
If f and g are continuous on $[a,b]$ and differentiable on (a,b), then there is c in (a,b) such that

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c). \tag{5.7}$$

Proof. Consider $H(x) = Ag(x) + Bf(x)$, where $A = f(b) - f(a)$, $B = g(a) - g(b)$. It is easily verified that $H(a) = H(b)$. Hence $H'(c) = 0$ for some c in (a,b). Thus

$$Af'(c) + Bg'(c) = 0,$$

as required. ∎

Theorem 5.10 The Mean Value Theorem
If f is continuous on $[a,b]$ and differentiable on (a,b), then there is c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Put $g(x) = x$ in the statement of Cauchy's Mean Value Theorem. ∎

5.12 Exercises

5.1 Directly from the definition, show that

$$\frac{d}{dx}(cy) = c \cdot \frac{dy}{dx}.$$

5.2 Let $y = a^x$, $a > 0$. Use the chain rule to find $\frac{dy}{dx}$.
Hint:

$$a^x = e^{x \log a}.$$

5.3 Let $y = x^x$, $x > 0$. Find $\frac{dy}{dx}$.

5.4 Let $u = \arctan x$. Show that

$$(1+x^2)\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = 0.$$

Differentiate n-times using Leibniz's theorem and hence determine the values of $u^{(n)}(0), \quad n \geq 1.$

5.5 If u_n denotes the nth derivative of $\dfrac{Lx+M}{x^2-2Bx+C}$, show that

$$\frac{x^2-2Bx+C}{(n+1)(n+2)}u_{n+2} + \frac{2(x-B)}{n+1}u_{n+1} + u_n = 0.$$

Hint: First derive the equation for $n = 0$, then use Leibniz's theorem.

5.6 If $y = (\arcsin x)^2$, and y_n denotes the nth derivative of y, show that

$$(1-x^2)y_{n+2} - 2x(n+1)y_{n+1} - n(n+1)y_n = 0, n \geq 1.$$

Hence determine $y_n(0), \quad n \geq 1.$

5.7 Show that

$$n(1-x^{-1/n}) < \log x < n(x^{1/n}-1), \quad \text{for} \quad x > 1, \;\; n \geq 1.$$

Hence show that

$$\left(1+\frac{y}{n}\right)^n < e^y < \left(1-\frac{y}{n}\right)^{-n}.$$

5.8 Show that

$$\log x = \lim_{n\to+\infty} n\left(x^{1/n}-1\right), \quad x > 0.$$

5.9 If $a < b$ and $0 < \alpha < 1$, show that

$$a^\alpha b^{1-\alpha} < \alpha a + (1-\alpha)b,$$

as follows:

1. Write the expression as

$$b^{1-\alpha} - a^{1-\alpha} < (1-\alpha)(b-a)a^{-\alpha},$$

and use the Mean Value Theorem.

2. Alternatively, notice that $y = \log x$ is concave and, hence, derive the inequality.

5.10 Use the results of the previous question to derive the following inequality (Hölder's inequality):

For $p,\ q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $a_1, a_2, \ldots, a_n;\;\; b_1, b_2, \ldots, b_n > 0$,

$$\sum_{m=1}^{n} a_m b_m \leq \left(\sum_{m=1}^{n} a_m^p\right)^{1/p} \cdot \left(\sum_{m=1}^{n} b_m^q\right)^{1/q}.$$

5.11 From the results of the previous question, derive the following inequality (Minkowski's inequality):

If $p > 1$ and $a_i,\ b_i > 0, \quad i = 1, 2, \ldots, n$, show that

$$\left[\sum_{m=1}^{n} (a_m + b_m)^p\right]^{1/p} \leq \left(\sum_{m=1}^{n} a_m^p\right)^{1/p} + \left(\sum_{m=1}^{n} b_m^p\right)^{1/p}.$$

5.12 **(a)** Using **5.9**, show that if $p,\ q > 0$ are such that $p + q = 1$, then, for any positive a_1, a_2 :

$$pa_1 + qa_2 < a_1^p a_2^q,$$

unless $a_1 = a_2$, in which case $pa_1 + qa_2 = a_1^p a_2^q$.

(b) By induction, show that if $a_1, a_2, \ldots, a_n > 0$ and $p_1 + p_2 + \cdots p_n = 1$, $p_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, n$, then, unless all a_i are equal,

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} < p_1 a_1 + p_2 a_2 + \cdots p_n a_n.$$

(c) From **(a)** and **(b)** prove the geometric-arithmetic means inequality for positive numbers a_i:

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n},$$

where equality takes place if and only if all $a_i, \quad i = 1, 2, \ldots, n$, are equal.

Chapter 6

Applications of the Mean Value Theorem

6.1 Taylor's Theorem

Theorem 6.1 Taylor's Theorem I *Suppose that f is $(n+1)$-times continuously differentiable in an open interval (a,b) and let x_o be any point of (a,b). Let $P_{n,\,x_o}(x)$ be the Taylor polynomial of degree n for the function f about the point x_o:*

$$P_{n,\,x_o}(x) = f(x_o) + \frac{f'(x_o)}{1}(x-x_o) + \frac{f''(x_o)}{2!}(x-x_o)^2 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n, \quad x \in (a,b),$$

and denote by $R_n(x)$ the remainder

$$R_n(x) = f(x) - P_{n,\,x_o}(x).$$

Then, for each $x \in (x_o, b)$, there exists some ξ, $x_o < \xi < x$, that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_o)^{n+1}. \tag{6.1}$$

Proof. We prove this theorem by applying Rolle's theorem to an auxiliary function. Let x be an arbitrary point of (a,b). Consider the function

$$\Phi(t) = f(x) - P_{n,\,t}(x), \quad x_o \le t \le x,$$

and note that

$$\Phi(x_o) = f(x) - P_{n,\,x_o}(x) = R_n(x). \tag{6.2}$$

Clearly, $\Phi(t)$ is differentiable in the interval (x_o, x) and

$$\begin{aligned}\Phi'(t) &= -f'(t) - \sum_{k=1}^{n} \frac{d}{dt}\left[\frac{(x-t)^k}{k!} f^{(k)}(t)\right] \\ &= -f'(t) - \sum_{k=1}^{n}\left[-\frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) + \frac{(x-t)^k}{k!} f^{(k+1)}(t)\right].\end{aligned}$$

Thus, in calculating $\Phi'(t)$, we obtain $2n+1$ terms, and they all cancel out but one, to give

$$\Phi'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t). \tag{6.3}$$

Now we define the function

$$\Psi(t) = \Phi(t) - \left(\frac{x-t}{x-x_o}\right)^{n+1} \Phi(x_o), \quad x_o \le t \le x,$$

and assert that Ψ satisfies the hypotheses of Rolle's theorem on the interval $[x_o, x]$.

Clearly, Ψ is continuous on $[x_o, x]$, differentiable on (x_o, x) and

$$\Psi(x_o) = \Phi(x_o) - \left(\frac{x-x_o}{x-x_o}\right)^{n+1} \Phi(x_o) = 0,$$
$$\Psi(x) = \Phi(x) = f(x) - P_{n,\,x}(x) = f(x) - f(x) = 0.$$

By Rolle's theorem, $\Psi'(\xi) = 0$ for some point ξ, $x_o < \xi < x$.

Now,

$$\Psi'(t) = \Phi'(t) + \frac{(n+1)(x-t)^n}{(x-x_o)^{n+1}} \Phi(x_o), \quad x_o < t < x.$$

By substituting in the values for $\Phi'(t)$ and $\Phi(x_o)$ given by (6.2) and (6.3) and replacing t by ξ, we obtain

$$0 = \Psi'(\xi) = -\frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi) + \frac{(n+1)(x-\xi)^n}{(x-x_o)^{n+1}} R_n(x)$$

which gives the desired result (6.1). ∎

The approximation of functions by polynomials is a very useful technique in the analysis of real functions.

If f is n times continuously differentiable at a specified point $x = x_o$, then f can be approximated by the n-th Taylor polynomial $P_{n,\,x_o}(x)$ and the measure of the approximation is given by the remainder

$$R_n(x) = f(x) - P_{n,x_o}(x)$$

that can be expressed in the form specified by (6.1).

Example 6.1 *Find the Taylor polynomial for the function $f(x) = (1+x)^n$, where n is an integer, about the point $x_o = 0$.*

Solution. We have

$$f^{(0)}(x) = (1+x)^n,$$

$$f^{(1)}(x) = n(1+x)^{n-1},$$

$$f^{(2)}(x) + n(n-1)(1+x)^{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(n)}(x) = n(n-1)\cdots(n-(n-1)) = n!$$

$$f^{(m)}(x) = 0, \quad \text{for} \quad m \ge n+1.$$

Hence, at the point $x = 0$,

$$f^{(0)}(0) = 1$$

$$f^{(1)}(0) = n$$

$$f^{(2)}(0) = n(n-1)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f^{(n)}(0) = n!$$

$$f^{(m)}(0) = 0, \quad \text{for} \quad m \geq n+1.$$

The Taylor polynomial for $f(x) = (1+x)^n$ with $x_o = 0$ is

$$P_{n,0}(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots 1}{n!}x^n,$$

and the remainder is

$$R_n(x) \equiv 0.$$

Hence, by Taylor's theorem, we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n,$$

the familiar binomial formula. ∎

Example 6.2 *Find the Taylor polynomial for $f(x) = e^x$ about $x_o = 0$.*

Solution. We have $f^{(n)}(x) = e^x$ for all $n \geq 0$, so that $f^{(n)}(0) = 1, \quad n = 1, 2, \ldots$. Therefore the Taylor polynomial is

$$P_{n,\,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

The remainder is

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}e^{\xi}, \quad 0 < \xi < x.$$

By Taylor's theorem, therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\xi},$$

for some ξ between 0 and x.

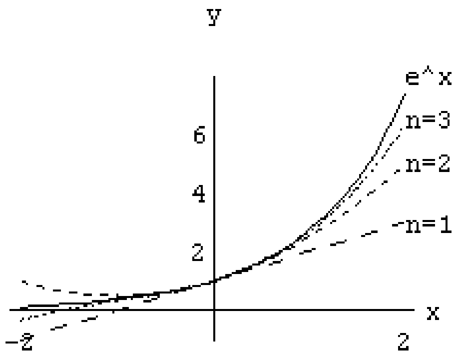
We note that $\left|\frac{x^{n+1}}{(n+1)!}e^{\xi}\right| \leq \frac{|x|^{n+1}}{(n+1)!} \cdot e^{|x|}$ so that, for fixed x, the remainder $R_n(x)$ tends to zero as n increases. Figure 6.1 shows how the polynomials $P_{n,0}$ approximate the function $f(x) = e^x$ on the intervals $(-2, 2)$, $(-3, 3)$, and $(-4, 4)$, when $n = 1, 2, 3, 4, 5, 6$.

In particular, if $x = 1$, we get

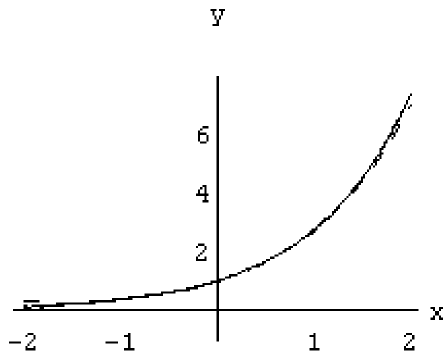
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^{\theta}}{(n+1)!},$$

where $0 < \theta < 1$. ∎

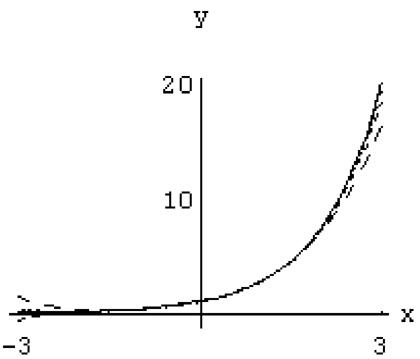
(a) $P_{n,0}(x)$, $n = 1, 2, 3$; $x \in (-2, 2)$



(b) $P_{n,0}(x)$, $n = 4, 5, 6$; $x \in (-2, 2)$



(c) $P_{n,0}(x)$, $n = 4, 5, 6$; $x \in (-3, 3)$



(d) $P_{n,0}(x)$, $n = 4, 5, 6$; $x \in (-4, 4)$

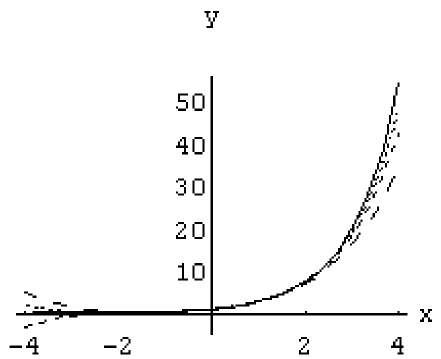


Figure 6.1: Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = e^x$.

Example 6.3 *Find the Taylor polynomial of degree* $2k+1$ *for the function* $f(x) = \sin x$ *with* $x_o = 0$.

Solution. If $f(x) = \sin x$ then $f^{(n)}(x) = \sin(x + n\pi/2)$, $n = 1, 2, \ldots$. Hence

$$f^{(2k)}(0) = \sin k\pi = 0, \quad \text{and} \quad f^{(2k+1)}(0) = \sin(k\pi + \pi/2) = (-1)^k, \;\; k = 1, 2, \ldots$$

The Taylor polynomial is

$$P_{2k+1,\,0}(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

An application of Taylor's theorem gives

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots (-1)^{k-1}\frac{x^{2k+1}}{(2k+1)!} + R_{2k+1},$$

where

$$R_{2k+1} = \frac{f^{(2k+2)}(\xi)}{(2k+2)!}x^{2k+2} = (-1)^{k+1}\frac{\sin\xi}{(2k+2)!}x^{2k+2}, \;\; 0 < \xi < x,$$

since

$$f^{(2k+2)}(\xi) = \sin[\xi + (2k+2)\pi/2] = \sin[\xi + (k+1)\pi] = (-1)^{k+1}\sin\xi.$$

Figure 6.2 shows Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \sin x$ on the interval $(-2\pi, 2\pi)$ for selected values of n. $\blacksquare$

(a) $P_{n,0}(x)$, $n = 1, 3, 5$; $x \in (-2\pi, 2\pi)$ **(b)** $P_{n,0}(x)$, $n = 7, 9, 11$; $x \in (-2\pi, 2\pi)$

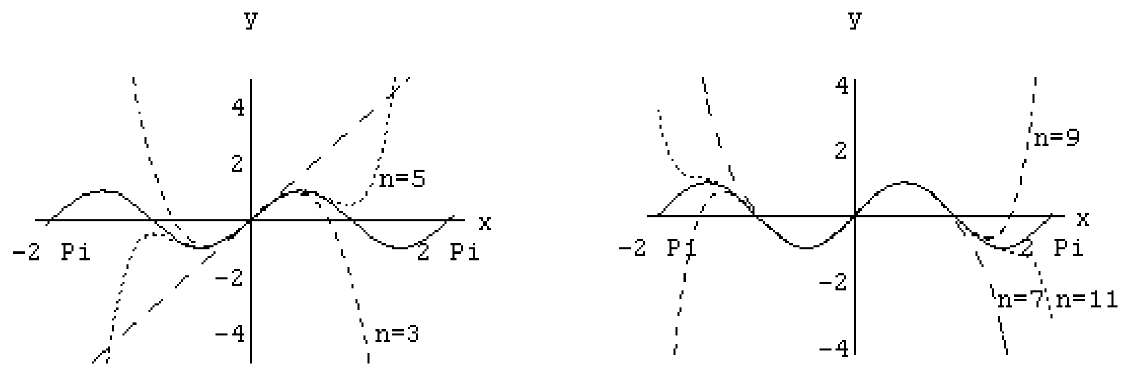


Figure 6.2: Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \sin x$.

Example 6.4 *Find the Taylor polynomial of degree* $2k$ *for the function* $f(x) = \cos x$ *with* $x_o = 0$.

Solution. If $f(x) = \cos x$ then $f^{(n)}(x) = \cos(x + n\pi/2)$, $n = 1, 2, \ldots$ Hence

$$f^{(2k)}(0) = \cos k\pi = (-1)^k, \quad \text{and} \quad f^{(2k+1)}(0) = \cos(k\pi + \pi/2) = 0, \;\; k = 1, 2, \ldots$$

The Taylor polynomial is

$$P_{2k,\,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^k\frac{x^{2k}}{(2k)!}.$$

Hence we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots (-1)^k\frac{x^{2k}}{(2k)!} + R_{2k},$$

where

$$R_{2k} = \frac{f^{(2k+1)}(\xi)}{(2k+1)!}x^{2k+1} = (-1)^{k+1}\frac{\cos\xi}{(2k+1)!}x^{2k+1}, \quad 0 < \xi < x,$$

since

$$f^{(2k+1)}(\xi) = \cos[\xi + (2k+1)\pi/2] = \cos(\xi + k\pi + \pi/2) = (-1)^{k+1}\cos\xi.$$

Figure 6.3 shows Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \cos x$ on the interval $(-2\pi, 2\pi)$ for selected values of n. ■

(a) $P_{n,0}(x)$, $n = 2, 4, 6$; $x \in (-2\pi, 2\pi)$ **(b)** $P_{n,0}(x)$, $n = 8, 10, 12$; $x \in (-2\pi, 2\pi)$

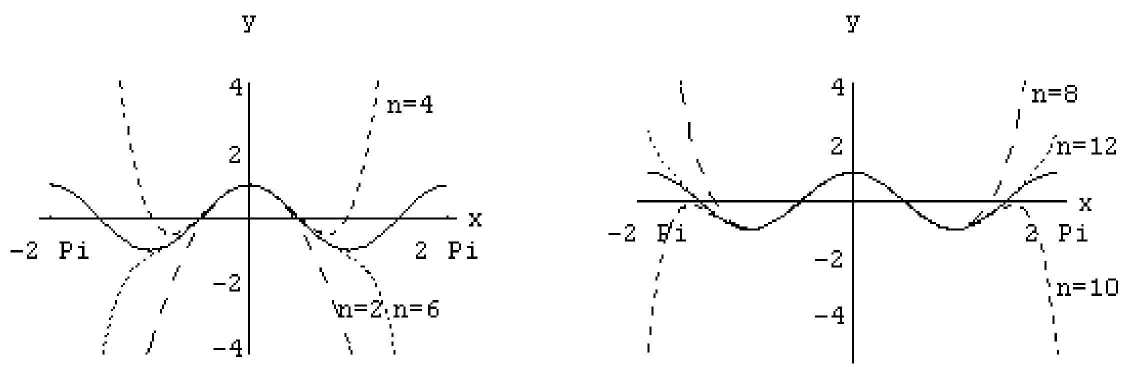


Figure 6.3: Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \cos x$.

Example 6.5

Find the Taylor polynomial of degree n for:

(a) $f(x) = \log x$ with $x_o = 1$, **(b)** $g(x) = \log(1 + x)$ with $x_o = 0$.

Solution.

(a) We have

$$f(x) = \log x,$$
$$f^{(1)}(x) = x^{-1},$$
$$f^{(2)}(x) = -x^{-2},$$
$$f^{(3)}(x) = 2x^{-3}$$
$$f^{(4)}(x) = -1 \cdot 2 \cdot 3x^{-4}$$
$$f^{(5)}(x) = 1 \cdot 2 \cdot 3 \cdot 4x^{-5}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus, we have

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k},$$

$$f^{(k)}(1) = (-1)^{k-1}(k-1)!, \quad k = 1, 2, \ldots$$

Hence

$$a_0 = f(1) = \log 1 = 0, \quad a_k = \frac{f^{(k)}(1)}{k!} = (-1)^{k-1}\frac{1}{k}, \quad k = 1, 2, \ldots$$

Hence the Taylor polynomial for $f(x) = \log x$ with $x_o = 1$ is

$$P_{n,\,1}(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \frac{(-1)^{n-1}}{n}(x-1)^n.$$

(b) To find the Taylor polynomial of degree n for the function $g(x) = \log(1+x)$ with $x_o = 0$ we note that

$$g(x) = f(1+x)$$

. Hence

$$g^{(k)}(x) = f^{(k)}(1+x), \quad g^{(k)}(0) = f^{(k)}(0), \quad k = 1, 2, \ldots$$

and the Taylor polynomial for $g(x) = \log(1+x)$ with $x_o = 0$ is

$$P_{n,\,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \frac{(-1)^{n+1}}{n}x^n.$$

Figure 6.4 shows Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \log(1+x)$ on the interval $(-1, 1)$ for selected values of n. ∎

(a) $P_{n,0}(x), \quad n = 1, 2, 3; \quad x \in (-1, 1)$ **(b)** $P_{n,0}(x), \quad n = 4, 5, 6; \quad x \in (-1, 1)$

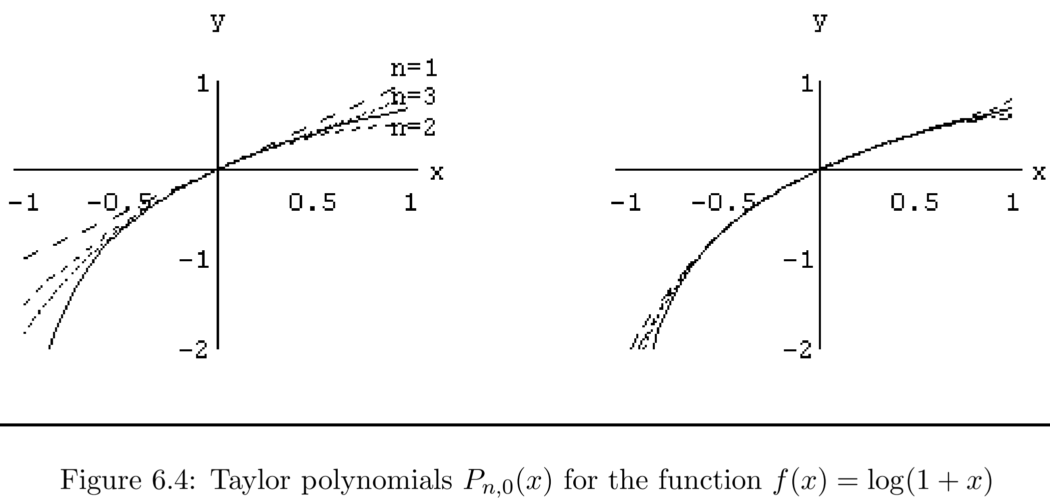


Figure 6.4: Taylor polynomials $P_{n,0}(x)$ for the function $f(x) = \log(1+x)$

6.2 Indeterminate Forms

In this section we consider several limits which take the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 1^{\infty}, \quad 0^{\infty}, \quad \infty^{0}.$$

To each of these symbols there corresponds an expression that involves two functions, say $f(x)$ and $g(x)$, and the limit, as $x \to x_o$, or as $x \to \infty$, of the expression considered.

Suppose that the functions f and g are defined in a (deleted) neighbourhood of a given point x_o and suppose that

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} g(x) = 0.$$

Then the limit

$$\lim_{x \to x_o} \frac{f(x)}{g(x)}$$

is said to be of the $\frac{0}{0}$ form. We have already considered some limits of this form, for example $\lim_{x \to 0} \frac{\sin x}{x}$.

The symbol $\frac{0}{0}$ is called an indeterminate symbol. This is because if $\frac{p}{q} = c \neq 0$, then c is uniquely determined by $p = q \cdot c$. When $p = q = 0$, any value of c satisfies $p = q \cdot c$.

Now we define the first two indeterminate symbols.

Definition 6.1 *The expression*

$$\frac{f(x)}{g(x)} \tag{6.4}$$

is of $\frac{0}{0}$ *form at the point* x_o*, if*

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} g(x) = 0;$$

it is of $\frac{0}{0}$ *form, as* $x \to \infty$*, if*

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0.$$

The expression (6.4) *is of* $\frac{\infty}{\infty}$ *form at the point* x_o*, if*

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} g(x) = \infty;$$

it is of $\frac{\infty}{\infty}$ *form, as* $x \to \infty$*, if*

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty.$$

Refer to Figure 6.5 to see several examples of undeterminate expressions $\dfrac{f(x)}{g(x)}$ at the point $x = 0$. Although all the expressions are of $\dfrac{0}{0}$ form, the limits $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$ are different. Example 6.8 deals with the evaluation of these limits.

A collection of theorems, called L'Hôpital rules, is useful in the evaluation of $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ forms at a given point $x = x_o$ and as $x \to \infty$.

We shall prove L'Hôpital rules for limits of the $\dfrac{0}{0}$ form at a given point $x = x_o$ and show how to manipulate other indeterminate forms through examples.

6.3 L'Hôpital's Rules

Theorem 6.2 L'Hôpital Rule I *Suppose that*

$$\lim_{x\to x_o} f(x) = 0 \ \textit{and} \ \lim_{x\to x_o} g(x) = 0,$$

and suppose that $\lim\limits_{x\to x_o}\dfrac{f'(x)}{g'(x)}$ *exists. Then* $\lim\limits_{x\to x_o}\dfrac{f(x)}{g(x)}$ *also exists and*

$$\lim_{x\to x_o}\frac{f(x)}{g(x)} = \lim_{x\to x_o}\frac{f'(x)}{g'(x)}. \tag{6.5}$$

Proof. The hypothesis that $\lim\limits_{x\to x_o}\dfrac{f'(x)}{g'(x)}$ exists implies that there is a (deleted) neighbourhood of x_o,

$$\mathcal{N}_{x_o,\delta} = \{x : 0 < |x - x_o| < \delta\}$$

such that

(i) f and g are continuous and differentiable on $\mathcal{N}_{x,\delta}$,

(ii) $g'(x) \neq 0$ for $x \in \mathcal{N}_{x,\delta}$.

Thus f and g are continuous on $(x_o - \delta,\ x_o + \delta)$ except, perhaps, for $x = x_o$, where f and g are not even assumed to be defined.

Step 1. We replace the functions f and g by F and G, respectively, where

$$F(x) = \begin{cases} f(x), & x \neq x_o \\ 0, & x = x_o, \end{cases}$$

$$G(x) = \begin{cases} g(x), & x \neq x_o \\ 0, & x = x_o. \end{cases}$$

The new functions F and G are continuous at the point x_o and consequently are continuous on the interval $(x_o - \delta,\ x_o + \delta)$. Also,

$$G'(x) \neq 0 \text{ for } x \in (x_o - \delta,\ x_o + \delta), \quad x \neq x_o. \tag{6.6}$$

Step 2. Let x be any point with $x_o < x < x_o + \delta$ and consider the interval (x_o, x). The functions F and G are continuous and differentiable on $[x_o, x]$. We apply the Mean Value Theorem to G to conclude that there exists x_1 in (x_o, x) such that

$$G'(x_1) = \frac{G(x) - G(x_o)}{x - x_o} = \frac{G(x)}{x - x_o}.$$

We see that

$$G(x) \neq 0,$$

for if $G(x) = 0$ there would be $G'(x_1) = 0$, where $x_1 \in (x_o - \delta,\ x_o + \delta)$, contradicting (6.6).

Step 3. Now we apply the Cauchy Mean Value Theorem to F and G on the interval $[x_o, x]$ to conclude that there is a number ξ in (x_o, x) such that

$$[F(x) - F(x_o)]G'(\xi) = [G(x) - G(x_o)]F'(\xi)$$

or

$$F(x)G'(\xi) = G(x)F'(\xi)$$

which gives

$$\frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)}. \tag{6.7}$$

The number ξ obviously depends on x. Since $\xi \in (x_o, x)$, we see that $\xi \to x_o+$ as $x \to x_o+$. Since $\lim_{x \to x_o} \frac{F'(x)}{G'(x)}$ exists, the one-sided limits of $\frac{F'(x)}{G'(x)}$, as $x \to x_o$, exist and are equal. By (6.7) we obtain

$$\lim_{x \to x_o+} \frac{F(x)}{G(x)} = \lim_{x \to x_o+} \frac{F'(\xi)}{G'(\xi)} = \lim_{\xi \to x_o+} \frac{F'(\xi)}{G'(\xi)}.$$

Hence

$$\lim_{x \to x_o+} \frac{F(x)}{G(x)} = \lim_{x \to x_o} \frac{F'(x)}{G'(x)}. \tag{6.8}$$

Step 4. Let x be any number with $x_o - \delta < x_o$ and consider the interval $[x, x_o]$. Repeat the arguments of **Step 2** and **Step 3** to arrive at the conclusion that

$$\lim_{x \to x_o-} \frac{F(x)}{G(x)} = \lim_{x \to x_o} \frac{F'(x)}{G'(x)}. \tag{6.9}$$

Step 5. Combining (6.8) and (6.9), we conclude that $\lim_{x \to x_o} \frac{F(x)}{G(x)}$ exists and

$$\lim_{x \to x_o} \frac{F(x)}{G(x)} = \lim_{x \to x_o} \frac{F'(x)}{G'(x)}$$

which is equivalent to the required statement (6.5). ∎

Example 6.6 *Evaluate* $A = \lim_{x\to 0} \frac{e^x - e^{-x}}{\log(e-x) + (x-1)}$.

Solution.

$$f(x) = e^x - e^{-x}, \quad g(x) = \log(e-x) + (x-1),$$
$$f'(x) = e^x + e^{-x}, \quad g'(x) = 1 - \frac{1}{e-x},$$
$$g'(0) \neq 0.$$

Hence

$$A = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{e^x + e^{-x}}{1 - \frac{1}{e-x}} = \frac{2e}{e-1}. \quad \blacksquare$$

Example 6.7 *Evaluate* $A = \lim_{x\to 1} \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$.

Solution.

$$f(x) = \sqrt{2x - x^4} - \sqrt[3]{x}, \qquad g(x) = 1 - \sqrt[4]{x^3},$$
$$f'(x) = \frac{1 - 2x^3}{\sqrt{2x - x^4}} - \frac{1}{3\sqrt[3]{x^2}}, \qquad g'(x) = -\frac{3}{4\sqrt[4]{x}}.$$

Since $f'(1) = -\frac{4}{3}$, $\quad g'(1) = -\frac{3}{4} \neq 0$, we obtain

$$A = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \frac{-4/3}{-3/4} = \frac{16}{9}. \quad \blacksquare$$

Example 6.8 *Using L'Hôpital's Rule to evaluate limits of the* $\frac{0}{0}$ *form at a given point* $x = x_o$. **Note:** Figure 6.5 shows us graphs of the first six functions $\frac{f(x)}{g(x)}$ considered in this example.

(a) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$

(b) $\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5\cos x = 5.$

(c) $\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} = -\lim_{x\to 0} \frac{2\sin x \cos x}{2x} = -\lim_{x\to 0} \frac{\sin 2x}{2x} = -1.$

(d) $\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{1}{1/\cos^2 x} = \lim_{x\to 0} \cos^2 x = 1.$

(e) $\lim_{x\to 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x\to 0} \frac{\sin x}{2\sin x \cos x} = \lim_{x\to 0} \frac{1}{2\cos x} = \frac{1}{2}.$

(f) $\lim_{x\to 0} \frac{1 - \cos x^2}{\sin^2 x} = \lim_{x\to 0} \frac{2x \sin x^2}{2\sin x \cos x} = \lim_{x\to 0} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0} \sin x^2 = 1 \cdot 0 = 0.$

(g) $\lim_{x\to 0} \frac{2^x - 3^x}{\sin x} = \lim_{x\to 0} \frac{2^x \log 2 - 3^x \log 3}{\cos x} = \log 2 - \log 4 = \log \frac{2}{3}.$

(h) $\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^h}{1} = 1;$ **(i)** $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}.$

Example 6.9 *Evaluate*
$$A = \lim_{x\to 0} \frac{\sin x - x\cos x}{x \sin x}.$$

Solution. Let $f(x) = \sin x - x\cos x, \quad g(x) = x\sin x$. We see that $f(0) = g(0) = 0$, so that the limit is of $\frac{0}{0}$ form. We have

$$\begin{aligned} f'(x) &= \cos x - \cos x + x\sin x = x\sin x, \\ g'(x) &= \sin x + x\cos x. \end{aligned}$$

Since $f'(0) = g'(0) = 0$, the limit $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ is also of the $\frac{0}{0}$ form. We evaluate

$$\begin{aligned} f''(x) &= \sin x + x\cos x \\ g''(x) &= \cos x + \cos x - x\sin x = 2\cos x - x\sin x, \end{aligned}$$

and find out that $\lim\limits_{x\to 0} \dfrac{f''(x)}{g''(x)}$ exists. Namely,

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{\sin x + x\cos x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Thus, $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ exists which implies that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}$ exists. We obtain

$$A = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = 0. \quad \blacksquare$$

Example 6.10 *Repeated application of L'Hôpital Rule to limits of the* $\dfrac{0}{0}$ *form at a given point* $x = x_o$.

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{2x} = \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}.$

(b) $\lim\limits_{x\to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 2x^2 + x} = \lim\limits_{x\to 1} \dfrac{3x^2 - 2x - 1}{3x^2 - 4x + 1} = \lim\limits_{x\to 1} \dfrac{6x-2}{6x-4} = 2.$

(c) $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2} = \lim\limits_{x\to 0} \dfrac{2\sin x\cos x}{2x} = \lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} = -\lim\limits_{x\to 0} \dfrac{2\cos 2x}{2} = 1. \quad \blacksquare$

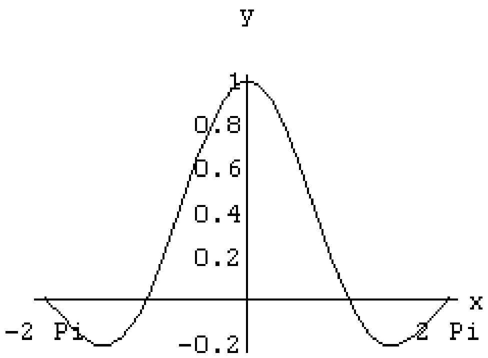
In Example 6.10, we needed to apply the L'Hôpital Rule twice for each of the limits. We must remember to verify the hypotheses of the rule each time we apply it. Consider the following use of the L'Hôpital Rule:

$$\lim_{x\to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x\to 1} \frac{6x}{2} = 3.$$

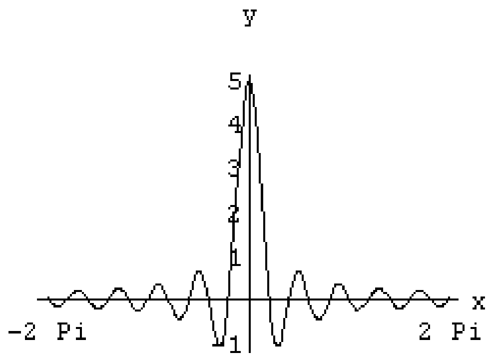
Note that $3x^2 + 1 \to 4$ and $2x - 3 \to -1$ when $x \to 1$, so that $\lim\limits_{x\to 1} \dfrac{3x^2+1}{2x-3}$ is not of the $\dfrac{0}{0}$ form and the L'Hôpital Rule is not applicable to this limit. In fact, we have

$$\lim_{x\to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \frac{4}{-1} = -4. \quad \blacksquare$$

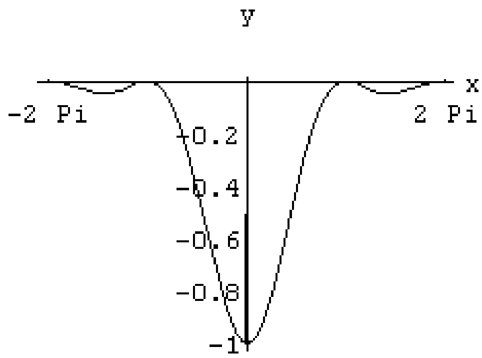
(a) $\dfrac{f(x)}{g(x)} = \dfrac{\sin x}{x}$



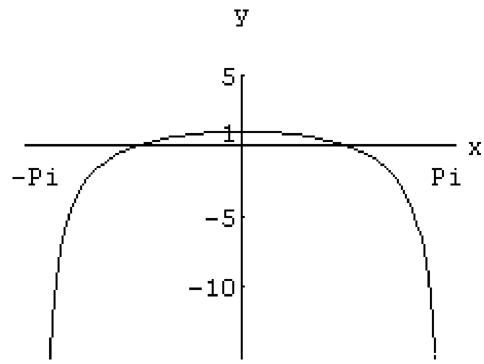
(b) $\dfrac{f(x)}{g(x)} = \dfrac{\sin 5x}{x}$



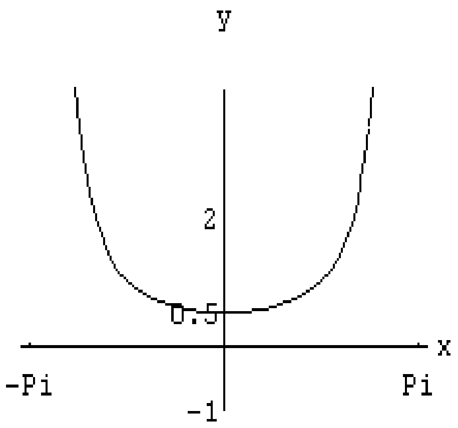
(c) $\dfrac{f(x)}{g(x)} = \dfrac{\cos^2 x - 1}{x^2}$



(d) $\dfrac{f(x)}{g(x)} = \dfrac{x}{\tan x}$



(e) $\dfrac{f(x)}{g(x)} = \dfrac{1 - \cos x}{\sin^2 x}$



(f) $\dfrac{f(x)}{g(x)} = \dfrac{1 - \cos x^2}{\sin^2 x}$

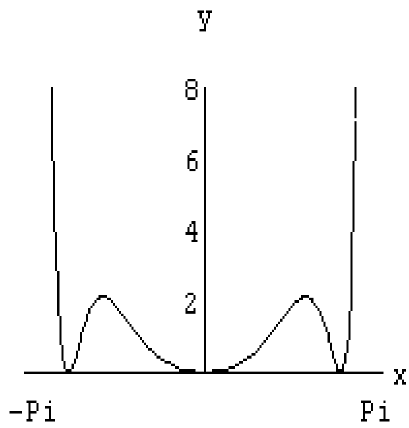


Figure 6.5: Functions $\dfrac{f(x)}{g(x)}$ that are of $\dfrac{0}{0}$ form at the point $x = 0$.

Example 6.11 *Show that we cannot use l'Hôpital's Rule to evaluate*

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}$$

which is of $\frac{0}{0}$ form at $x=0$.

Solution. Let $f(x)=x^2\sin\dfrac{1}{x}$ and $g(x)=\sin x$. Then $f'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$ and $g'(x)=\cos x$. Now

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x},$$

which does not exist as $\lim\limits_{x\to 0}\cos\dfrac{1}{x}$ does not exist.

However,

$$\lim_{x\to 0}\frac{x}{\sin x}=1,\qquad \lim_{x\to 0}x\sin\frac{1}{x}=0.$$

Hence

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}x\sin\frac{1}{x}=1\cdot 0=0. \quad \blacksquare$$

By repeated application of the L'Hôpital Rule I we obtain the following generalization of the rule.

Theorem 6.3 Generalization of L'Hôpital Rule for limits of the $\frac{0}{0}$ form *Let f and g be n-times continuously differentiable on the interval $[a,b]$ and suppose that*

$$f^{(r)}(x_o)=g^{(r)}(x_o)=0 \ \ \textit{for}\ \ r=0,1,2,\ldots,n-1,$$

where $x_o\in(a,b)$. If $g^{(n)}(x_o)\neq 0$ and $\lim\limits_{x\to x_o}\dfrac{f^{(n)}(x_o)}{g^{(n)}(x_o)}$ exists, then $\lim\limits_{x\to x_o}\dfrac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_o}\frac{f(x)}{g(x)}=\lim_{x\to x_o}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

Example 6.12 *Evaluate*

$$A=\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}.$$

Solution. We have

$$\begin{array}{llll}
f(x)=e^x-e^{-x}-2x, & g(x)=x-\sin x, & f(0)=0, & g(0)=0,\\
f'(x)=e^x+e^{-x}-2, & g'(x)=1-\cos x, & f'(0)=0, & g'(0)=0,\\
f''(x)=e^x-e^{-x}, & g''(x)=\sin x, & f''(0)=0, & g''(0)=0,\\
f^{(3)}(x)=e^x+e^{-x}, & g^{(3)}(x)=\cos x, & f^{(3)}(0)=2, & g^{(3)}(0)=1.
\end{array}$$

Hence

$$A = \lim_{x\to 0} \frac{f^{(3)}(x)}{g^{(3)}(x)} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{2}{1} = 2. \quad \blacksquare$$

Theorem 6.4 L'Hôpital Rule II *Assume that f and g are differentiable (and hence continuous) on a given interval $[b, +\infty)$, that is for sufficiently large x. Assume further that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $g'(x) \neq 0$ for all $x > b$.*

If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l.$$

Proof. We assume that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l$, so that

$$\forall \varepsilon > 0 \ \exists a \ \left(x > a \implies \left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon \right) \tag{6.10}$$

Given $\varepsilon_1 > 0$ we show that $\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon_1$ for sufficiently large values of x.

Let ε_1 be given. We can assume that $g(x) > g(a)$ for every $x > a$. Otherwise we get contradiction with the assumption that $g'(x) \neq 0$. Explicitly, suppose that $g(x_1) \le g(a)$ for some $x_1 \in (a, \infty)$. Since $\lim_{x\to\infty} g(x) = \infty$, there is $x_2 > x_1$ such that $g(x_2) \ge g(a)$. By the Intermediate Value Theorem, we have $g(c) = g(a)$ for some c, $x_1 \le c \le x_2$. By Rolle's Theorem, there is d between a and c, such that $g'(d) = 0$.

Now, with any $x > a$ we consider the interval $[a, x]$ and apply the Cauchy Mean Value Theorem to the functions f and g on this interval to get

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x_o)}{g'(x_o)}, \quad \text{for some } \ x_o \in (a, x).$$

Thus, using (6.10) with $\varepsilon = \frac{\varepsilon_1}{2}$, we conclude that for all $x > a$,

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - l \right| < \frac{\varepsilon_1}{2}. \tag{6.11}$$

Since $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, we can choose $x > a$ so large that $f(x) > f(a)$ and $g(x) > 0$. We can assume, therefore, that for sufficiently large x,

$$g(x) > g(a), \quad f(x) > f(a), \quad g(x) > 0. \tag{6.12}$$

We can write

$$\left| \frac{f(x)}{g(x)} - l \right| \le \left| \frac{f(x)}{g(x)} - \frac{f(x) - f(a)}{g(x) - g(a)} \right| + \left| \frac{f(x) - f(a)}{g(x) - g(a)} - l \right|. \tag{6.13}$$

Now,

$$\left| \frac{f(x)}{g(x)} - \frac{f(x) - f(a)}{g(x) - g(a)} \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} \right| \cdot \left| \frac{f(x)}{f(x) - f(a)} \frac{g(x) - g(a)}{g(x)} - 1 \right|. \tag{6.14}$$

By (6.11), we have

$$l - \frac{\varepsilon_1}{2} < \frac{f(x)-f(a)}{g(x)-g(a)} < l + \frac{\varepsilon_1}{2} \quad \Longrightarrow \quad \left|\frac{f(x)-f(a)}{g(x)-g(a)}\right| < |l| + \frac{\varepsilon_1}{2}.$$

Since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, we have

$$\lim_{x\to\infty} \frac{f(x)}{f(x)-f(a)} \, \frac{g(x)-g(a)}{g(x)} = 1,$$

which implies that, if x is sufficiently large, then

$$\left|\frac{f(x)}{f(x)-f(a)} \, \frac{g(x)-g(a)}{g(x)} - 1\right| < \varepsilon_2,$$

where $\varepsilon_2 = \dfrac{\varepsilon_1}{2(|l|+\varepsilon_1/2)}$. Hence (6.14) implies that

$$\left|\frac{f(x)}{g(x)} - \frac{f(x)-f(a)}{g(x)-g(a)}\right| < \left(|l| + \frac{\varepsilon_1}{2}\right) \cdot \frac{\varepsilon_1}{2(|l|+\varepsilon_1/2)} = \frac{\varepsilon_1}{2}.$$

Consequently, using (6.11) and (6.13) we get

$$\left|\frac{f(x)}{g(x)} - l\right| \le \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1$$

for all sufficiently large values of x. This completes the proof. ∎

(a) $\dfrac{f(x)}{g(x)} = \dfrac{x}{e^x}$ **(b)** $\dfrac{f(x)}{g(x)} = \dfrac{\tan x}{\tan 3x}$

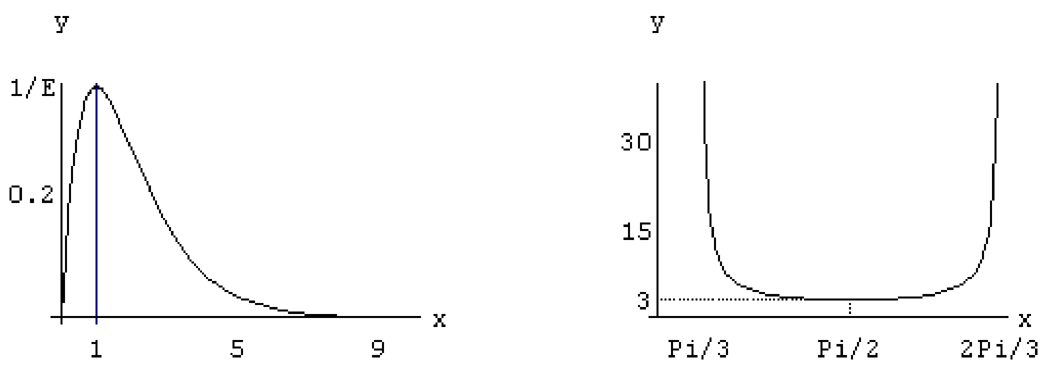


Figure 6.6: Functions $\dfrac{f(x)}{g(x)}$ that are of $\dfrac{\infty}{\infty}$ form, **(a)** as $x \to \infty$, **(b)** as $x \to \dfrac{\pi}{2}$.

Example 6.13 *Find* $\lim_{x\to\infty} \frac{x}{e^x}$.

Solution. Let $f(x) = x, \quad g(x) = e^x$, so that $f'(x) = 1, \quad g'(x) = e^x$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty, \quad \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Since $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ also exists and we have

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Figure 6.6(a) shows us a graph of the function $\frac{f(x)}{g(x)} = \frac{x}{e^x}$ over the interval $(0, 10)$. ∎

(a) $\frac{f(x)}{g(x)} = \frac{\log x}{x^\alpha}, \quad \alpha = 0.5;$ **(b)** $\frac{f(x)}{g(x)} = \frac{\log x}{x^\alpha}, \quad \alpha = 1.1;$ **(c)** $\frac{f(x)}{g(x)} = \frac{\log x}{x^\alpha}, \quad \alpha = 1.5;$

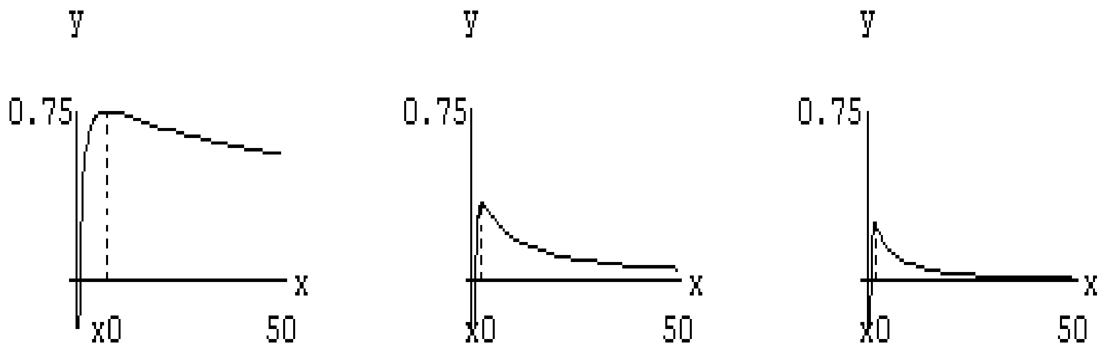


Figure 6.7: Illustrating the concept of $\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0$ for selected values of α, $\alpha > 0$; x0=$e^{1/\alpha}$.

Example 6.14 *Using L'Hôpital's Rule to evaluate limits of the* $\frac{\infty}{\infty}$ *form.*

(a) $$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\tan 3x} = \lim_{x\to\frac{\pi}{2}} \frac{1/\cos^2 x}{3/\cos^2 3x} = \frac{1}{3}\left(\lim_{x\to\frac{\pi}{2}} \frac{\cos 3x}{\cos x}\right)^2 = \frac{1}{3}\left(\lim_{x\to\frac{\pi}{2}} \frac{3\sin 3x}{\sin x}\right)^2 = \frac{1}{3}\cdot 9 = 3.$$

(b) $$\lim_{x\to\infty} \frac{\log x}{x^\alpha} = \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0, \quad \text{for} \quad \alpha > 0.$$ ∎

6.4 Further Indeterminate Forms

We shall illustrate the methods through specific examples. The general pattern should be clear.

Example 6.15 *Limits of the* $\infty - \infty$ *form.*

(a) Evaluate $A = \lim_{x\to\infty}[x - \sqrt{x^2+x}]$.

We have

$$x - \sqrt{x^2+x} = x\left(1 - \sqrt{1+\frac{1}{x}}\right) = \frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}},$$

which is the $\frac{0}{0}$ form. Hence

$$\begin{aligned} A &= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\left[-\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\right] = -\frac{1}{2}. \end{aligned}$$

(b) Evaluate $A = \lim_{x\to 1}\left(\frac{1}{\log x} - \frac{1}{x-1}\right)$.

$$\begin{aligned} A &= \lim_{x\to 1}\left(\frac{1}{\log x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\log x}{(x-1)\log x} = \lim_{x\to 1}\frac{1-1/x}{\log x + 1 - 1/x} \\ &= \lim_{x\to 1}\frac{1/x^2}{1/x+1/x^2} = \frac{1}{2}. \quad \blacksquare \end{aligned}$$

Example 6.16 *Limits of the* $0\cdot\infty$ *form.*

(a) $\lim_{x\to 0+}(x\log x) = \lim_{x\to 0+}\frac{\log x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0.$

(b) $\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{\cot}{1/x} = \lim_{x\to 0}\frac{-1/\sin^2 x}{-1/x^2} = \left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2 = 1^2 = 1.$

(c) $\lim_{x\to 0+} x^\alpha \log x = \lim_{x\to 0+}\frac{\log x}{x^{-\alpha}} = \lim_{x\to 0+}\frac{1/x}{-\alpha x^{-\alpha-1}} = -\frac{1}{\alpha}\lim_{x\to 0+} x^\alpha = 0, \quad \alpha > 0.$ $\blacksquare$

Example 6.17 *Limit of the* 0^0 *form.*

Evaluate $A = \lim_{x\to 0+}(\sin x)^x$.

We note that $\log(\sin x)^x = x\log(\sin x) = \frac{\log(\sin x)}{1/x}$ is of the $\frac{\infty}{\infty}$ form. Let $f(x) = \log(\sin x)$ and $g(x) = 1/x$. Then

$$\frac{f'(x)}{g'(x)} = \frac{\cos x/\sin x}{-1/x^2} = -x\cos x \cdot \frac{x}{\sin x} \to 0, \quad \text{as} \quad x \to 0+.$$

Hence

$$A = \lim_{x\to 0+} (\sin x)^x = \exp\left(\lim_{x\to 0+} \log(\sin x)^x\right) = \exp(0) = 1. \quad \blacksquare$$

Example 6.18 *Limit of the ∞^0 form.*

$$\lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x} = \lim_{x\to 0} \exp(-\tan x \log x^2) = e^0 = 1,$$

for

$$\lim_{x\to 0}(\tan x \log x^2) = \lim_{x\to 0} \frac{2\log x}{\cot x} = \lim_{x\to 0} \frac{2/x}{-1/\sin^2 x} = \lim_{x\to 0} \frac{-2\sin^2 x}{x} = 0. \quad \blacksquare$$

Example 6.19 *Limits of the 1^∞ form.*

(a) Evaluate $A = \lim_{x\to 0}(1-2x)^{3/x}$.

We have

$$(1-2x)^{3/x} = \exp\left(3 \cdot \frac{\log(1-2x)}{x}\right),$$

$$\lim_{x\to 0} \frac{\log(1-2x)}{x} = \lim_{x\to 0} \frac{-\frac{2}{1-2x}}{1} = -2.$$

Hence

$$A = \exp\left(\lim_{x\to 0} \frac{\log(1-2x)}{x}\right) = e^{3\cdot(-2)} = e^{-6}.$$

(b) Evaluate $A = \lim_{x\to 1} x^{\frac{1}{1-x}}$.

We have

$$x^{\frac{1}{1-x}} = \exp\left(\frac{\log x}{1-x}\right),$$

$$\lim_{x\to 1} \frac{\log x}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1.$$

Hence

$$A = \exp\left(\lim_{x\to 1} \frac{\log x}{1-x}\right) = e^{-1} = \frac{1}{e}. \quad \blacksquare$$

(a) $x\log(1+\dfrac{1}{x}) \to 1, \text{ as } x\to\infty$ **(b)** $(1+\dfrac{1}{x})^x \to e, \text{ as } x\to\infty$

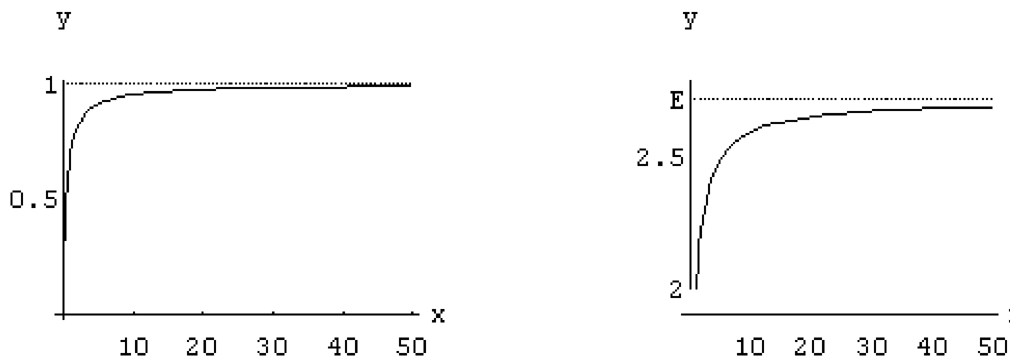


Figure 6.8: Illustrating convergence of the functions $x\log(1+\frac{1}{x})$ and $f(x)=(1+\frac{1}{x})^x$, as $x\to\infty$.

Example 6.20 *Evaluate* $A = \lim\limits_{x\to\infty}(1+\dfrac{1}{x})^x$.

Solution. We have

$$(1+\frac{1}{x})^x = e^{\log(1+\frac{1}{x})^x} = e^{x\log(1+\frac{1}{x})}.$$

Now evaluate $B = \lim\limits_{x\to\infty} x\log(1+\dfrac{1}{x}) = \lim\limits_{x\to\infty}\dfrac{\log(1+\frac{1}{x})}{\frac{1}{x}}$:

$$\left(\log(1+\tfrac{1}{x})\right)' = \left(1+\tfrac{1}{x}\right)^{-1}\left(-\tfrac{1}{x^2}\right)$$
$$\left(\tfrac{1}{x}\right)' = -\tfrac{1}{x^2}.$$

Hence

$$B = \lim_{x\to\infty} x\log(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-1} = 1.$$

Finally, we use the continuity of e^x to conclude that

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{x\to\infty} e^{x\log(1+\frac{1}{x})} = e^B = e^1 = e.$$

Refer to Figure 6.8 which illustrates the convergence of the functions $x\log(1+\frac{1}{x})$ and $(1+\frac{1}{x})^x$ to 1 and e, respectively, as $x\to\infty$. ∎

6.5 Monotone Functions

Theorem 6.5 *Suppose that f is differentiable (and hence continuous) in a given interval (a,b) and $f'(x) = 0$ for all $x \in (a,b)$. Then f is a constant function on (a,b):*

$$f(x) = C, \quad x \in (a,b),$$

for some value of C.

Proof. Let $x_1,\ x_2$ ba any two points in $(a,\ b)$. By the Mean Value Theorem, there is c between x_1 and x_2 such that

$$f(x_1) - f(x_2) = (x_1 - x_2)\ f'(c).$$

By assumption, $f'(c) = 0$. Hence $f(x_1) = f(x_2)$ for any $x_1,\ x_2 \in (a,\ b)$, which implies that f is a constant function on $(a,\ b)$. $\blacksquare$

If the derivatives of two functions f and g are equal, then applying the above Theorem to $f - g$ we conclude that $f - g = C$, for some constant C.

Corollary 6.1 *If f and g are differentiable in (a,b) and*

$$f'(x) = g'(x), \quad x \in (a,b),$$

then there is a constant C such that

$$f(x) = g(x) + C, \quad x \in (a,b).$$

Example 6.21 *Showing that the identity $f(x) \equiv g(x)$ holds on a given interval.*

We shall prove the following identity:

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}. \tag{6.15}$$

Let

$$f(x) = \arctan x, \quad g(x) = \arcsin \frac{x}{\sqrt{1+x^2}}.$$

We have

$$f'(x) = \frac{1}{1+x^2},$$

$$g'(x) = \frac{1}{\sqrt{1 - \dfrac{x^2}{1+x^2}}} \cdot \frac{\sqrt{1+x^2} - \dfrac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{1+x^2}.$$

Since $f'(x) = g'(x)$ for $-\infty < x < \infty$, by Corollary 6.1, there is a constant C such that the identity

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} + C$$

holds for x in any finite interval (a, b) and therefore for every x in $(-\infty, \infty)$. To evaluate C, let $x = 0$. Then $f(0) = \arctan 0 = 0$ and $g(0) = \arcsin 0 = 0$ imply that $C = 0$. Hence the identity (6.15) is proved. ∎

Theorem 6.6 *Suppose that f is differentiable on $[a, b]$ and continuous on $[a, b]$.*

(a) *If $f'(x) > 0$ for all x in (a, b), then f is strictly increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for all x in (a, b), then f is strictly decreasing on $[a, b]$.*

Proof. Let $x_1,\ x_2 \in [a, b]$ and let $x_1 < x_2$. By the Mean Value Theorem we can write

$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2), \quad \text{for some} \quad \xi \in (x_1, x_2).$$

(a) By the hypothesis, $f'(\xi) > 0$. Thus

$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) < 0 \quad \Longrightarrow \quad f(x_1) < f(x_2).$$

(b) By the hypothesis, $f'(\xi) < 0$. Thus

$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) > 0 \quad \Longrightarrow \quad f(x_1) > f(x_2).$$ ∎

Example 6.22 *Showing that the inequality $f(x) \ge g(x)$ holds in a given interval.*

We shall apply Theorem 6.6 to prove the following inequalities:

(a) $(1 + x)^a \ge 1 + ax, \quad x > 0, \ a \ge 1.$

(b) $\sin x < x, \ x > 0.$

(c) $\sin x \ge x - \dfrac{x^3}{6}, \ x \ge 0.$

Solution.

(a) Let $f(x) = (1 + x)^a - (1 + ax)$. Then $f'(x) = a(1 + x)^{a-1} - a = a[(1 + x)^{a-1} - 1] > 0$ for $x > 0, \ a \ge 1$.

Hence $f(x)$ is increasing on $[0, \ \infty)$, which implies that

$$f(x) > f(0) = 0 \ \text{ for } \ x > 0.$$

(b) Let $f(x) = x - \sin x$. Then $f'(x) = 1 - \cos x \ge 0$, for all x.

Hence $f(x)$ is increasing on $(-\infty, \ \infty)$, and

$$f(x) \ge f(0) = 0 \ \text{ for } \ x > 0.$$

(c) Let $f(x) = \sin x - x + \dfrac{x^3}{6}$. Then $f'(x) = \cos x - 1 + \dfrac{x^2}{2}$ and $f''(x) = -\sin x + x, \ \ x \ge 0.$

Since $f''(x) \ge 0$, for $x \ge 0$, we conclude that $f'(x)$ is increasing on $[0, \ \infty)$ which implies that

$$f'(0) \ge f'(0) = 0, \ \text{ for } \ x \ge 0.$$

Hence we conclude that $f(x)$ is increasing on $[0, \ \infty]$ and consequently

$$f(x) > f(0) = 0, \ \text{ for } \ x > 0.$$

∎

6.6 Convex and Concave Functions

A function f with domain $\mathcal{D}$ is convex on a given interval I (contained in $\mathcal{D}$), if the line segment joining any two points P$=(x_1,\ f(x_1))$ and Q$=(x_2,\ f(x_2))$ on the curve $y=f(x)$, $x_1,\ x_2 \in I$, lies above the graph of the function f. If this line segment lies below the graph of f, then the function f is said to be concave.

Figure 6.9: Illustrating the concept of a convex function.

Refer to Figure 6.9 and note that the equation of the line through the two points P$=(x_1,\ f(x_1))$ and Q$=(x_2,\ f(x_2))$ is

$$y=\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1)=g(x).$$

Thus, f is convex if $f(x)\le g(x)$ for $x_1<x<x_2$ and f is concave on I if $f(x)\ge g(x)$ for $x_1<x<x_2$, where $x_1,\ x_2$ are any points in I. Now,

$$\begin{aligned} f(x)\le g(x) \quad &\iff \quad f(x)\le \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1)\\ &\iff \quad \frac{f(x_2)-f(x_1)}{x_2-x_1}\ge\frac{f(x)-f(x_1)}{x-x_1}. \qquad (6.16)\end{aligned}$$

The inequality (6.16) can be written equivalently using a parameter t, $0\le t\le 1$. Note that if $x_1<x_2$, then as t ranges from 0 to 1, the point $x=(1-t)x_1+tx_2$ traverses the interval (x_1,x_2) and the point (x,y), where $y=(1-t)f(x_1)+tf(x_2)$, traverses the line segment joining the points P and Q.

Hence (6.16) is equivalent to

$$f[(1-t)\ x_1+t\ x_2]\le(1-t)\ f(x_1)+t\ f(x_2), \qquad (6.17)$$

or

$$f[s\ x_1+t\ x_2]\le s\ f(x_1)+t\ f(x_2), \qquad (6.18)$$

whenever $s+t=1,\quad 0\le s\le 1,\quad 0\le t\le 1.$

Definition 6.2 *A function f with domain $\mathcal{D}$ is* **convex** *on a given interval $I \in \mathcal{D}$, if for any $x_1, x_2, x \in I$ such that $x_1 < x < x_2$ the following condition holds:*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x) - f(x_1)}{x - x_1}.$$

The function f is **concave** *on I, if*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x) - f(x_1)}{x - x_1}$$

for any $x_1, x_2, x \in I$ such that $x_1 < x < x_2$.

We note that the above definition does not require a convex/concave function to be differentiable at all. If f is differentiable, it is convex on the interval I, if and only if its derivative f' is increasing on I. If f is twice differentiable on I, therefore f is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.

Theorem 6.7 *Suppose that a function f with domain $\mathcal{D}$ is differentiable on $I = [a, b] \subset \mathcal{D}$.*
(a) *The function f is convex on I if and only if f' is increasing on I.*

(b) *The function f is concave on I if and only if f' is decreasing on I.*

Proof. We note that the condition (6.16) can be rewritten equivalently as

$$(x_2 - x)f(x_1) + (x_1 - x_2)f(x) + (x - x_1)f(x_2) \geq 0,$$

or

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x) - f(x_2)}{x - x_2}, \tag{6.19}$$

where $x_1 < x < x_2,\ \ x_1,\ x_2 \in I$.

$\Longrightarrow$ We assume that f is convex on I, so that (6.19) holds for any choice of $x_1 < x < x_2$ in I. Since f is differentiable on I, $f'(x_1)$ and $f'(x_2)$ exist. Using (6.19), we have

$$f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

and

$$f'(x_2) = \lim_{x \to x_2} \frac{f(x) - f(x_2)}{x - x_2} \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus

$$f'(x_1) \leq \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq f'(x_2).$$

Hence, if $x_1,\ x_2 \in I$ and $x_1 < x_2$, then $f'(x_1) \leq f'(x_2)$ so f' is increasing on I.

$\Longleftarrow$ We assume that f' is increasing on I. Let $x_1,\ x_2,\ x \in I$ and $x_1 < x < x_2$. Applying the Mean Value Theorem to f on $[x_1, x]$ gives

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(\xi_1) \text{ for some } \xi_1, \ \ x_1 < \xi_1 < x.$$

Applying the Mean Value Theorem to f on $[x, x_2]$ gives

$$\frac{f(x_2) - f(x)}{x_2 - x} = f'(\xi_2) \text{ for some } \xi_2, \ \ x < \xi_2 < x_2.$$

Now, f' is increasing on I, so that

$$\xi_1 < \xi_2 \quad \Longrightarrow \quad f'(\xi_1) \le f'(\xi_2) \quad \Longrightarrow \quad \frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x) - f(x_2)}{x - x_2}.$$

Hence the proof of Part (a) of the theorem is complete.

To prove Part (b), we observe that if $g(x) = -f(x)$ is convex with increasing derivative $g'(x) = -f'(x)$, then $-g(x) = f(x)$ is concave with decreasing derivative. Thus application of the above proof to the function $g(x) = -f(x)$ proves Part (b) of the theorem. ∎

Suppose now that f is twice differentiable on I. Then f' is increasing on I if and only if $f''(x) \ge 0, \ \ x \in I$, and f' is decreasing on I, if and only if$f''(x) \le 0, \ \ x \in I$. Hence we obtain the following theorem.

Theorem 6.8 *Let I be an open interval and suppose that $f : I \mapsto \mathbb{R}$ has a second derivative.*

(a) *The function f is convex on I if and only if $f''(x) \ge 0$, for all x in I.*

(b) *The function f is concave on I if and only if $f''(x) \le 0$, for all x in I.*

6.7 Partial Derivatives

Consider a real-valued function $F(x, y)$ defined in a neighbourhood of the point (x_o, y_o). The partial derivative of $F(x, y)$ with respect to x at the point (x_o, y_o) is denoted by one of the symbols

$$\frac{\partial F}{\partial x}(x_o, y_o), \quad \frac{\partial F(x_o, y_o)}{\partial x}, \quad F_x(x_o, y_o)$$

and calculated simply by differentiating $F(x, y)$ with respect to x, treating y as a constant.

Similarly, the partial derivative of $F(x, y)$ with respect to y at the point (x_o, y_o), denoted by one of the symbols,

$$\frac{\partial F}{\partial y}(x_o, y_o), \quad \frac{\partial F(x_o, y_o)}{\partial y}, \quad F_y(x_o, y_o)$$

is calculated by differentiating $F(x, y)$ with respect to y, treating x as a constant.

Definition 6.3 *Let $F(x,y)$ be a real-valued function defined in a neighbourhood of the point (x_o, y_o). The partial derivative of $F(x,y)$ with respect to x at the point (x_o, y_o) is*

$$F_x(x_o, y_o) = \lim_{h\to 0} \frac{F(x_o + h, y_o) - F(x_o, y_o)}{h},$$

provided that the above limit exists.

The partial derivative of $F(x,y)$ with respect to y at the point (x_o, y_o) is

$$F_y(x_o, y_o) = \lim_{k\to 0} \frac{F(x_o, y_o + k) - F(x_o, y_o)}{k},$$

provided that the above limit exists.

We can see that the above definition corresponds to the definition of an ordinary derivative for the function $f(x) = F(x, y_o)$. Hence, all the rules of differentiation of a function of one variable we have developed are valid for partial derivatives.

Example 6.23 *Finding partial derivatives.*

(a) If $F(x,y) = x^2 - y + 3y^2$, $(x,y) \in \mathbb{R}^2$, then the partial derivatives of $F(x,y)$ exist at any point (x,y) of $\mathbb{R}^2$:

$$F_x(x,y) = 2x, \quad F_y(x,y) = 6y - 1.$$

(b) If $F(x,y) = \dfrac{xy}{y-1}$, $y \neq 1$, then the partial derivatives of F exist at any point (x,y) of the domain $\mathcal{D}$ of F, where $\mathcal{D} = \{(x,y) : -\infty < x < \infty,\ \ y \neq 1\}$. We have

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{y}{y-1}, \\ \frac{\partial F}{\partial y} &= \frac{x(y-1) - xy}{(y-1)^2} = -\frac{x}{(y-1)^2}. \end{aligned}$$

(c) If $F(x,y) = \arctan \dfrac{y}{x}$, $x \neq 0$, then we have

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{-y/x^2}{1 + \left(\frac{y}{x}\right)^2} = -\frac{y}{x^2 + y^2}, \\ \frac{\partial F}{\partial y} &= \frac{1/x}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}, \end{aligned}$$

at any point (x,y) of the domain $\mathcal{D} = \{(x,y) : x \neq 0,\ \ -\infty < y < \infty\}$ of the function F.

(d) Let $F(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

If $(x, y) \neq (0, 0)$, then

$$F_x(x, y) = \frac{\partial}{\partial x}\left(\frac{2xy}{x^2 + y^2}\right) = 2\frac{y(x^2 + y^2) - 2x^2 y}{(x^2 + y^2)^2} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^2}.$$

To find $F_x(0, 0)$ we use the definition:

$$F_x(0, 0) = \lim_{h \to 0} \frac{F(h, 0) - F(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Hence

$$F_x(x, y) = \begin{cases} \dfrac{2y(y^2 - x^2)}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases} \blacksquare$$

6.8 The Implicit Function Theorem Revisited

Theorem 6.9 The Implicit Function Theorem II
Let F be a function of two variables x and y, where $x \in I$, $y \in J$, I, J are open intervals. Suppose that for $x \in I$, $y \in J$,

1. *$F(x, y)$ is continuous;*
2. *The partial derivatives F_x and F_y exist and are continuous;*
3. *For some point (x_o, y_o), $x_o \in I$, $y_o \in J$, we have $F(x_o, y_o) = 0$ and $F'_y(x_o, y_o) \neq 0$.*

Then there are positive numbers h and k that determine the rectangle

$$\mathcal{R} = \{(x, y) : \quad |x - x_o| < h, \ \ |y - y_o| < k\}$$

such that the equation

$$F(x, y) = 0$$

defines y as a function of x,

$$y = f(x), \ \ x \in I_o = \{x : |x - x_o| < h\} \subseteq I,$$

whose range is contained in

$$J_o = \{y : |y - y_o| < k\} \subseteq J.$$

The function f is determined uniquely on I and has the following properties:

(a) $f(x_o) = y_o$,

(b) *f is continuous on I_o,*

(c) *f is differentiable on I_o,*

(d) *f' is continuous on I_o and can be expressed as*

$$f'(x) = -\frac{F_x(x, y)}{F_y(x, y)}.$$

Lemma 6.1 *Suppose that the function $G(x, y)$ possesses the partial derivatives $G_x = G_x(x, y)$ and $G_y = G_y(x, y)$ at the point (x_o, y_o) and in a neighbourhood of (x_o, y_o). If G_x and G_y are continuous as functions of two variables x and y, then the difference*

$$\Delta G(x_o, y_o) = G(x_o + \Delta x, y_o + \Delta y) - G(x_o, y_o)$$

can be expressed as

$$\Delta G(x_o, y_o) = G_x(x_o, y_o)\ \Delta x + G_y(x_o, y_o)\ \Delta y + \alpha(\Delta x, \Delta y)\ \Delta x + \beta(\Delta x,\ \Delta y)\ \Delta y, \quad (6.20)$$

where $\alpha \to 0$ and $\beta \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$.

Proof. Clearly, we have

$$\Delta G(x_o, y_o) = [G(x_o + \Delta x, y_o + \Delta y) - G(x_o, y_o + \Delta y)] + [G(x_o, y_o + \Delta y) - G(x_o, y_o)].$$

By the Mean Value Theorem, there exists c_1 between x_o and $x_o + \Delta x$, $c_1 = x_o + \theta_1 \Delta x,\ \ 0 < \theta_1 < 1$, such that

$$G(x_o + \Delta x, y_o + \Delta y) - G(x_o, y_o + \Delta y) = G_x(x_o + \theta_1 \Delta x, y_o + \Delta y)\Delta x.$$

Similarly, there exists $c_2 = y_o + \theta_2 \Delta y, \quad 0 < \theta_2 < 1$, such that

$$G(x_o, y_o + \Delta y) - G(x_o, y_o) = G_y(x_o, y_o + \theta_2 \Delta y)\Delta y.$$

When $\Delta x \to 0$ and $\Delta y \to 0$ then $x_o + \theta_1 \Delta x \to x_o$ and $y_o + \theta_2 \Delta y \to y_o$. Since G_x and G_y are continuous, there exist $\alpha = \alpha(\Delta x, \Delta y)$ and $\beta = \beta(\Delta x, \Delta y)$, such that

$$G_x(x_o + \theta_1 \Delta x,\ y_o + \Delta y)\ \Delta x = G_x(x_o, y_o) + \alpha(\Delta x, \Delta y), \quad (6.21)$$

$$G_y(x_o,\ y_o + \theta_2 \Delta y)\ \Delta y = G_y(x_o, y_o) + \beta(\Delta x, \Delta y), \quad (6.22)$$

where $\alpha(\Delta x, \Delta y) \to 0$ and $\beta(\Delta x, \Delta y) \to 0$, when $\Delta x \to 0$ and $\Delta y \to 0$.

Hence, we obtain

$$\Delta G(x_o, y_o) = G_x(x_o, y_o)\Delta x + G_y(x_o, y_o)\Delta y + \alpha \Delta x + \beta \Delta y.$$

∎

Proof of the Implicit Function Theorem II. By the hypothesis 3, $F_y(x_o, y_o) \neq 0$. We assume that $F_y(x_o, y_o) > 0$; otherwise we replace F by $-F$ and repeat the argument.

Since F_y is continuous, there is a (sufficiently small) square

$$S = \{(x, y) : |x - x_o| \le k, |y - y_o| \le k\}$$

on which F is positive. For each fixed value $x = x^*,\ \ |x^* - x_o| < k$, we have $F_y(x^*, y) > 0$, which implies that $F(x, y)$ is increasing as a function of y for fixed value of x, $(x, y) \in \mathcal{D} = I \times J$, the domain of F. Therefore, the Implicit Value Theorem I is applicable.

The existence of a unique function $y = f(x)$ that is continuous on I_o and satisfies the property $f(x_o) = y_o$ is then guaranteed by that theorem. We are only to prove the properties **(c)** and **(d)** for the function f.

Let $x \in I_o$. For a given point (x, y) on the curve $y = f(x)$, consider another point $(x + \Delta x, y + \Delta y)$ on the same curve and define

$$\Delta F(x, y) = F(x + \Delta x, y + \Delta y) - F(x, y).$$

We have

$$\begin{aligned} y &= f(x), & y + \Delta y &= f(x + \Delta x), \\ F(x, y) &= 0, & F(x + \Delta x, y + \Delta y) &= 0, \end{aligned}$$

so that, for $(x, y) \in I \times J$,

$$\Delta F(x, y) = 0$$

Applying Lemma 6.1, we obtain

$$0 = \Delta F(x, y) = F_x(x, y)\Delta x + F_y(x, y)\Delta y + \alpha\Delta x + \beta\Delta y,$$

where $\alpha \to 0$ as $\Delta x \to 0$ and $\beta \to 0$ as $\Delta y \to 0$. This implies that

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x, y)}{\Delta x} = -\frac{F_x(x, y) + \alpha}{F_y(x, y) + \beta}$$

and gives

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} -\frac{F_x(x, y) + \alpha}{F_y(x, y) + \beta}.$$

Now, if $\Delta x \to 0$ then $\Delta y = f(x + \Delta x) - f(x) \to f(x) - f(x) = 0$, since f is continuous. Hence $\alpha \to 0$, $\beta \to 0$, and

$$\lim_{\Delta x \to 0} \frac{F_x(x, y) + \alpha}{F_y(x, y) + \beta} = -\frac{F_x(x, y)}{F_y(x, y)}$$

which implies that $f'(x)$ exists and is given by

$$f'(x) = -\frac{F_x(x, y)}{F_y(x, y)}, \quad x \in I. \blacksquare$$

6.9 Exercises

6.1 Evaluate, using L'Hôpital's rules, the following limits:

(i) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$;

(ii) $\lim_{x \to +\infty} \frac{x^a}{e^x}$;

6.2 Show that, for $0 < x < \pi/2$, the following functions are increasing functions:

(i) $\dfrac{x}{\sin x}$ **(ii)** $\dfrac{x^2/2}{1-\cos x}$ **(iii)** $\dfrac{x^3/6}{x-\sin x}$

6.3 From the previous question **(i)** deduce that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

6.4 Show that

$$\pi < \frac{\sin \pi x}{x(1-x)} \le 4, \quad \text{for} \quad 0 < x < 1.$$

Sketch the graph of the function.

6.5 Sketch the graph of the function

$$f(x) = \pi \cot \pi x - \frac{1}{x} - \frac{1}{x-1}, \quad 0 < x < 1.$$

6.6 Sketch the general form of the graph of y, given that

$$\frac{dy}{dx} = \frac{(6x^2+x-1)(x-1)^2(x+1)^3}{x^2}.$$

6.7 **(a)** Use Cauchy's Mean Value Theorem to show that there are numbers c, d, e, between x and a such that:

(i) $\dfrac{f(x)-f(a)}{x-a} = f^{(1)}(c)$,

(ii) $\dfrac{f(x)-[f(a)+(x-a)f^{(1)}(a)]}{(x-a)^2} = \dfrac{1}{2}\cdot f^{(2)}(d)$,

(iii) $\dfrac{f(x)-\left[f(a)+(x-a)f^{(1)}(a)+\dfrac{(x-a)^2}{2}f^{(2)}(a)\right]}{(x-a)^3} = \dfrac{1}{3!}\cdot f^{(3)}(e)$.

(b) Use part **(a)** to construct an inductive proof of Taylor's Theorem.

6.8 Determine if $F(x,y)$ has continuous partial derivatives F_x, F_y, F_{xy}, F_{yx}, F_{xx}, F_{yy}, where

$$F(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{when } (x,y) \ne (0,0) \\ 0, & \text{when } (x,y) = (0,0). \end{cases}$$

Is F continuous at $(0,0)$?

6.9 Show that $x + y + \sin xy = 0$ defines y as a function of x in $(-h,h)$, y in $(-k,k)$, for some $h,\ k > 0$.

Determine $\dfrac{dy}{dx}$.

Sketch the graph of y.

Chapter 7

Integration

7.1 Lower and Upper Sums

Definition 7.1 *A* **partition** $\mathcal{P}$ *of the closed interval* $[a, b]$ *is a finite collection of points*

$$\{x_o, x_1, x_2, \dots, x_n\}$$

that satisfy the condition

$$a = x_o < x_1 < x_2 < \cdots < x_n = b.$$

Definition 7.2 *Consider a function* f *that is bounded on an interval* $[a, b]$ *and let*

$$\mathcal{P} = \{x_o, x_1, x_2, \dots, x_n\}$$

be a partition of $[a, b]$*. Let*

$$\Delta x_i = x_i - x_{i-1}, \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x), \quad M_i = \sup_{x_{i-1} \le x \le x_i} f(x).$$

The **lower sum** *of* f *for* $\mathcal{P}$*, denoted by* $\mathcal{L}(\mathcal{P}, f)$*, is*

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

The **upper sum** *of* f *for* $\mathcal{P}$*, denoted by* $\mathcal{U}(\mathcal{P}, f)$*, is*

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

If f is a positive function, the lower and upper sums have a clear geometric interpretation. The lower sum $\mathcal{L}(\mathcal{P}, f)$ is the area of the union of rectangles with base $[x_{k-1}, x_k]$ and height m_k; the upper sum $\mathcal{U}(\mathcal{P}, f)$ is the area of the union of rectangles with base $[x_{k-1}, x_k]$ and height M_k (see Figure 7.1).

Example 7.1 *Consider the partition*

$$\mathcal{P} = \{x_o, x_1, x_2, \ldots, x_n\}, \quad x_i = \frac{i}{n}, \; i = 0, 1, \ldots, n$$

of the interval $I = [0, 1]$ *and consider the function* $f(x) = x^2$ *on the interval* I. *Find the lower sum and the upper sum of the function* f *for the partition* $\mathcal{P}$.

Solution. For the partition $\mathcal{P}$, we have

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} = \tfrac{i}{n} - \tfrac{i-1}{n} = \tfrac{1}{n}, \quad i = 1, 2, \ldots, n, \\
m_i &= \inf_{x_{i-1} \le x \le x_i} f(x) = f(x_{i-1}) = \left(\tfrac{i-1}{n}\right)^2, \quad i = 1, 2, \ldots, n, \\
M_i &= \sup_{x_{i-1} \le x \le x_i} f(x) = f(x_i) = \left(\tfrac{i}{n}\right)^2, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

The lower sum is

$$\begin{aligned}
\mathcal{L}(\mathcal{P}, f) &= m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots m_n \Delta x_n \\
&= \frac{1}{n}(m_1 + m_2 + \cdots + m_n) \\
&= \frac{1}{n}\left(\frac{0}{n^2} + \frac{1}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right) \\
&= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) \\
&= \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}.
\end{aligned}$$

The upper sum is

$$\begin{aligned}
\mathcal{U}(\mathcal{P}, f) &= M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots M_n \Delta x_n \\
&= \frac{1}{n}(M_1 + M_2 + \cdots + M_n) \\
&= \frac{1}{n}\left(\frac{1}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) \\
&= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\
&= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}. \quad \blacksquare
\end{aligned}$$

Definition 7.3 *Let* $\mathcal{P} = \{x_o, x_1, \ldots, x_n\}$ *be a partition of the interval* I. *If* $\mathcal{Q} = \{y_o, y_1, \ldots, y_m\}$, $m \ge n$, *is a partition of* I *such that each point* x_k, $1 \le k \le n$, *of the partition* $\mathcal{P}$ *belongs to* $\mathcal{Q}$, *that is if* $\mathcal{P} \subseteq \mathcal{Q}$, *then we say that the partition* $\mathcal{Q}$ *is a* **refinement** *of the partition* $\mathcal{P}$.

Figure 7.1 shows the lower and upper sums for $f(x) = x^2$ corresponding to three partitions of the interval $I = [0, 1]$; the partition $\mathcal{P} = \{0, \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, 1\}$ and two refinements $\mathcal{Q}_1 = \{0, \frac{1}{20}, \frac{2}{20}, \ldots, \frac{19}{20}, 1\}$ and $\mathcal{Q}_2 = \{0, \frac{1}{40}, \frac{2}{40}, \ldots, \frac{39}{40}, 1\}$ of the partition $\mathcal{P}$.

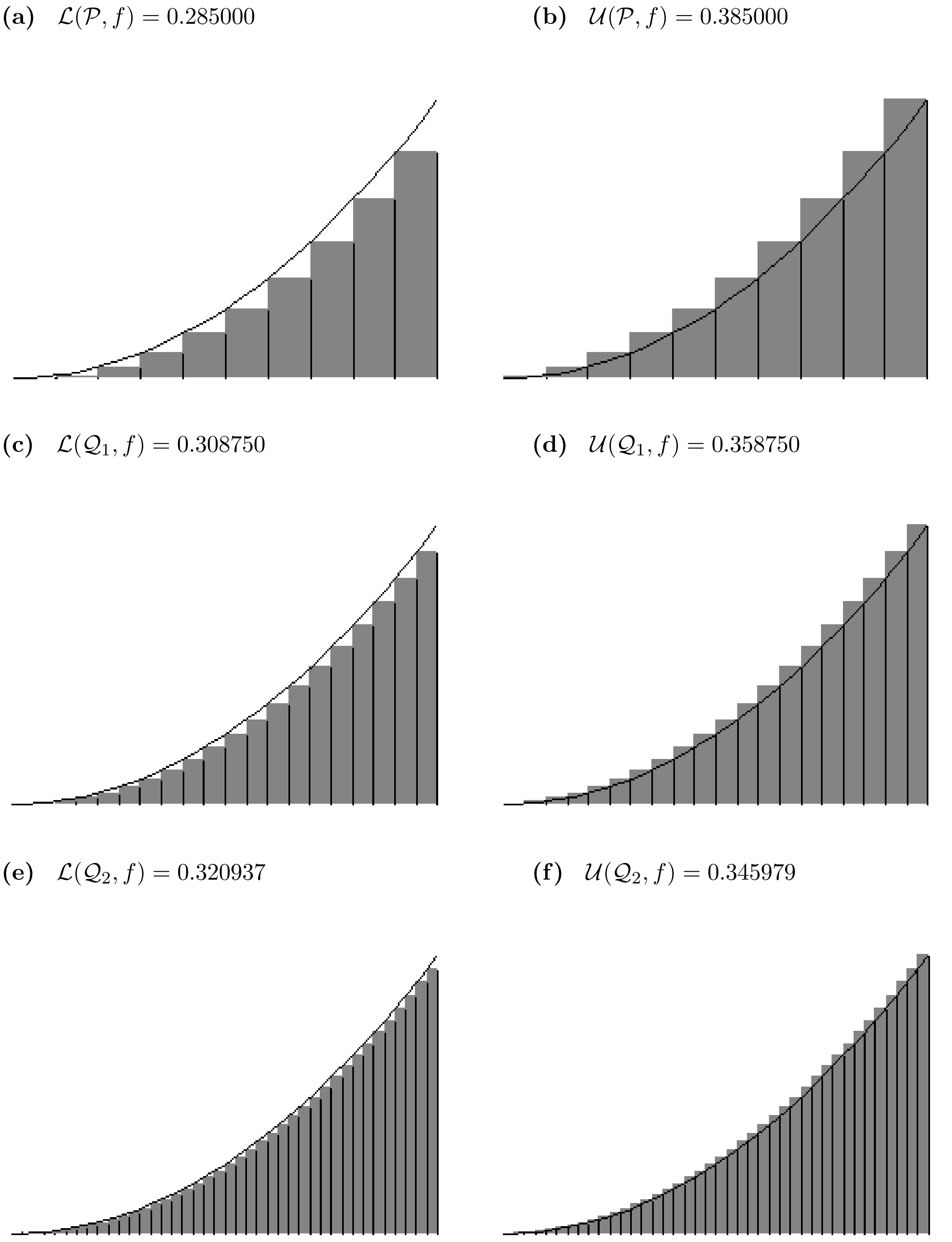


Figure 7.1: The lower and upper sums for $f(x) = x^2$ corresponding to the partition $\mathcal{P} = \{0, \frac{1}{10}, \frac{1}{20}, \ldots, 1\}$ and two refinements of $\mathcal{P}$.

Now we shall prove the following results which are clear on geometric grounds:

- the lower sum is less than or equal to the upper sum of the same partition;
- refining a partition increases lower sums and decreases upper sums.

We shall combine them to conclude that a lower sum is always less than or equal to an upper sum even if they correspond to different partitions.

Lemma 7.1 *Let f be a bounded function defined on a given interval $I = [a, b]$ and let $\mathcal{P} = \{x_o, x_1, \ldots, x_n\}$ be a partition of I. Denote by m and M the infimum and supremum of f on I:*

$$m = \inf_{x \in I} f(x), \quad M = \sup_{x \in I} f(x).$$

The following hold:

(a) $m(b-a) \leq \mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f) \leq M(b-a)$.

(b) *If $\mathcal{Q}$ is a refinement of the partition $\mathcal{P}$, then*

$$\mathcal{L}(\mathcal{P}, f) \leq \mathcal{L}(\mathcal{Q}, f) \quad \textit{and} \quad \mathcal{U}(\mathcal{Q}, f) \leq \mathcal{U}(\mathcal{P}, f).$$

Proof. Let

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x), \quad \Delta_k = x_k - x_{k-1}, \quad k = 1, 2, \ldots, n.$$

(a) We have

$$\mathcal{L}(\mathcal{P}, f) = \sum_{k=1}^{n} m_k \Delta_k, \quad \mathcal{U}(\mathcal{P}, f) = \sum_{k=1}^{n} M_k \Delta_k.$$

Since

$$m \leq m_k \leq M_k \leq M, \quad k = 1, 2, \ldots, n, \quad \textit{and} \quad \sum_{k=1}^{n} \Delta_k = b - a,$$

we get

$$m(b-a) = \sum_{k=1}^{n} m\Delta_k \leq \sum_{k=1}^{n} m_k \Delta_k \leq \sum_{k=1}^{n} M_k \Delta_k \leq \sum_{k=1}^{n} M\Delta_k = M(b-a),$$

and so

$$m(b-a) \leq \mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f) \leq M(b-a).$$

(b) If $\mathcal{Q}$ is a refinement of $\mathcal{P}$, $\mathcal{P} \subseteq \mathcal{Q}$, then $\mathcal{Q}$ can be obtained from $\mathcal{P}$ by adjoining a finite number of points to $\mathcal{P}$.

Let $\mathcal{P}'$ be the partition obtained from $\mathcal{P}$ by adjoining **one** point z to $\mathcal{P}$, where $x_{k-1} < z < x_k$, for some k:

$$\mathcal{P}' = \{x_o, x_1, \ldots, x_{k-1}, z, x_k, \ldots, x_n\}.$$

Let

$$m'_k = \inf_{x\in[x_{k-1},z]} f(x), \quad m''_k = \inf_{x\in[z,x_k]} f(x),$$

and let $\Delta'_k = z - x_{k-1}$ and $\Delta''_k = x_k - z$. Then $m_k \le m'_k$ and $m_k \le m''_k$, and we have

$$m_k\Delta_k = m_k\Delta'_k + m_k\Delta''_k \le m'_k\Delta'_k + m''_k\Delta''_k.$$

Hence

$$\mathcal{L}(\mathcal{P}', f) = \sum_{j\neq k} m_j\Delta_j + m'_k\Delta'_k + m''_k\Delta''_k \ge \sum_{j\neq k} m_j\Delta_j + m_k\Delta_k = \mathcal{L}(\mathcal{P}, f).$$

We can see, therefore, that adjoining one point to $\mathcal{P}$ increases the lower sum:

$$\mathcal{L}(\mathcal{P}', f) \ge \mathcal{L}(\mathcal{P}, f).$$

Since the refinement $\mathcal{Q}$ of the partition $\mathcal{P}$ can be obtained by adjoining a finite number of points to $\mathcal{P}$, one at a time, repeating the above argument we conclude that

$$\mathcal{L}(\mathcal{P}, f) \le \mathcal{L}(\mathcal{Q}, f).$$

Now we shall examine how the upper sum changes when one point is adjoined to the partition. Let

$$M'_k = \sup_{x\in[x_{k-1},z]} f(x), \quad M''_k = \sup_{x\in[z,x_k]} f(x).$$

Then $M_k \ge M'_k, \quad M_k \ge M''_k$ and

$$\mathcal{U}(\mathcal{P}', f) = \sum_{j\neq k} M_j\Delta_j + M'_k\Delta'_k + M''_k\Delta''_k \le \sum_{j=1}^{n} M_j\Delta_j = \mathcal{U}(\mathcal{P}, f).$$

Hence the upper sum decreases when a point is adjoined to the partition:

$$\mathcal{U}(\mathcal{P}', f) \le \mathcal{U}(\mathcal{P}, f).$$

Now, adjoining a finite number of points to $\mathcal{P}$ we obtain $\mathcal{Q}$, so repeating the above argument we infer that

$$\mathcal{U}(\mathcal{Q}, f) \le \mathcal{U}(\mathcal{P}, f).$$

The proof is complete. ∎

Lemma 7.2 *Let f be a bounded function defined on a given interval $I \subset \mathbb{R}$ and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two partitions of I. Then*

$$\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{U}(\mathcal{P}_2, f).$$

Proof. Let $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}_1 \subseteq \mathcal{Q}$ and $\mathcal{P}_2 \subseteq \mathcal{Q}$, so $\mathcal{Q}$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. By Lemma 7.1(a) we have

$$\mathcal{L}(\mathcal{Q}, f) \le \mathcal{U}(\mathcal{Q}, f)$$

and, by Lemma 7.1(b), we have

$$\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{L}(\mathcal{Q}, f)$$

$$\mathcal{U}(\mathcal{Q}, f) \le \mathcal{U}(\mathcal{P}_2, f)$$

Hence the required result follows:

$$\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{L}(\mathcal{Q}, f) \le \mathcal{U}(\mathcal{Q}, f) \le \mathcal{U}(\mathcal{P}_2, f). \quad ∎$$

7.2 Lower and Upper Integrals

Let f be a bounded function defined on the interval I. To each partition $\mathcal{P}$ of the interval I there correspond its lower sum $\mathcal{L}(\mathcal{P}, f)$ and its upper sum $\mathcal{U}(\mathcal{P}, f)$. If we consider the collection $\boldsymbol{\mathcal{P}}(I)$ of all possible partitions of the interval I, we obtain two sets of numbers: the set of lower sums

$$S_{\mathcal{L}} = \{l : \ l = \mathcal{L}(\mathcal{P}, f), \ \ \mathcal{P} \in \boldsymbol{\mathcal{P}}(I)\}$$

and the set of upper sums

$$S_{\mathcal{U}} = \{u : \ u = \mathcal{U}(\mathcal{P}, f), \ \ \mathcal{P} \in \boldsymbol{\mathcal{P}}(I)\}.$$

Since f is bounded, by Lemma 7.1, each set $S_{\mathcal{L}}$ and $S_{\mathcal{U}}$ is bounded, and as such possesses an infimum and supremum. The supremum of $S_{\mathcal{L}}$ is called the lower integral of f on I and the infimum of $S_{\mathcal{U}}$ is called the upper integral of f on I. The function f is said to be integrable over the interval I if the lower and upper integrals are equal.

Definition 7.4 *Let f be a bounded function defined on the interval I. Let $\boldsymbol{\mathcal{P}}(I)$ be the collection of all possible partitions of the interval I.*

The **lower integral** *of f on I, denoted by $L(f)$, is*

$$L(f) = \sup_{\mathcal{P} \in \boldsymbol{\mathcal{P}}(I)} \{l : \ l = \mathcal{L}(\mathcal{P}, f)\}.$$

The **upper integral** *of f on I, denoted by $U(f)$, is*

$$U(f) = \inf_{\mathcal{P} \in \boldsymbol{\mathcal{P}}(I)} \{u : \ u = \mathcal{U}(\mathcal{P}, f)\}.$$

Example 7.2 *Finding lower and upper integrals.*

(a) Let $f(x) = c$ be a constant function on the interval $[a, b]$. Clearly

$$\mathcal{U}(\mathcal{P}, f) = c(b - a), \quad \mathcal{L}(\mathcal{P}, f) = c(b - a).$$

Hence

$$L(f) = U(f) = c(b - a).$$

(b) Let f be defined on the interval $[0, 1]$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Then

$$L(f) = 0, \quad U(f) = 1 \quad \text{on} \quad [0, 1]. \ \blacksquare$$

Theorem 7.1 *If f is a bounded function defined on the interval $I \subset \mathbb{R}$, then the lower integral $L(f)$ and upper integral $U(f)$ of f on the interval I exist and satisfy the inequality*

$$L(f) \leq U(f). \tag{7.1}$$

Proof. The existence of $L(f)$ and $U(f)$ follows directly from the hypothesis that f is bounded.

To prove the inequality (7.1), we let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any partitions of the interval I. Then $\mathcal{L}(\mathcal{P}_1, f) \leq \mathcal{U}(\mathcal{P}_2, f)$. Since $\mathcal{P}_1$ is arbitrarily chosen from the collection of partitions $\boldsymbol{\mathcal{P}}(I)$, we conclude that the number $\mathcal{U}(\mathcal{P}_2, f)$ is an upper bound of the set $S_{\mathcal{L}}$ of the lower sums. Now, $L(f)$ is the supremum of $S_{\mathcal{L}}$, so

$$L(f) \leq \mathcal{U}(\mathcal{P}_2, f).$$

Because $\mathcal{P}_2$ is arbitrarily chosen, the above inequality implies that the number $L(f)$ is a lower bound for the set $S_{\mathcal{U}}$ of upper sums. Consequently the infimum of this set, $U(f)$, must satisfy the inequality

$$L(f) \leq U(f). \quad \blacksquare$$

7.3 The Riemann Integral

Definition 7.5 *Let f be a bounded function on the interval $I = [a, b]$. The function f is said to be* **Riemann integrable** *on I if the lower integral $L(f)$ and the upper integral $U(f)$ are equal.*

If f is integrable on I then the **Riemann integral** *of f on I is defined to be the common value of $L(f)$ and $U(f)$ and is denoted by*

$$\int_a^b f(x)dx \quad or \quad \int_a^b f.$$

In addition, we define

$$\int_a^a f(x)dx = 0 \quad and \quad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Example 7.3 *Using Definition 7.5.*

(a) Let $f(x) = c$ be a constant function on the interval $[a, b]$. Clearly

$$\mathcal{U}(\mathcal{P}, f) = c(b-a) = \mathcal{L}(\mathcal{P}, f) = c(b-a).$$

Hence f is integrable and

$$\int_a^b c dx = \int_a^b f(x)dx = L(f) = U(f) = c(b-a).$$

(b) Let f be defined on the interval $[0,1]$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Then we have

$$0 = L(f) \neq U(f) = 1$$

and we conclude that f is not integrable on the interval $[0,1]$. ∎

7.4 Existence of the Integral

The above definition of the Riemann integral does not provide an easy criterion to establish whether or not $\int_a^b f(x)dx$ exists. The next theorem gives us a necessary and sufficient condition for existence of the integral.

Theorem 7.2 Integrability Criterion
Let f be a bounded function defined on $I = [a,b]$. The function f is integrable on I if and only if for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of I such that

$$\mathcal{U}(\mathcal{P}_\varepsilon, f) - \mathcal{L}(\mathcal{P}_\varepsilon, f) < \varepsilon. \tag{7.2}$$

Proof.
$\Longrightarrow$ We assume that f is integrable on I, so $L(f) = U(f)$. Let $\varepsilon > 0$ be given. Let $\boldsymbol{\mathcal{P}}$ be the collection of all possible partitions of I.

Since $L(f) = \sup\limits_{\mathcal{P}\in\boldsymbol{\mathcal{P}}(I)} \{l : \ l = \mathcal{L}(\mathcal{P}, f)\}$, by definition of the supremum of a set, we have

$$\exists \mathcal{P}_1 \in \boldsymbol{\mathcal{P}} \quad \left(L(f) - \frac{\varepsilon}{2} < \mathcal{L}(\mathcal{P}_1, f) \right). \tag{7.3}$$

Since $U(f) = \inf\limits_{\mathcal{P}\in\boldsymbol{\mathcal{P}}(I)} \{u : \ u = \mathcal{U}(\mathcal{P}, f)\}$, by definition of the infimum of a set , we have

$$\exists \mathcal{P}_2 \in \boldsymbol{\mathcal{P}} \quad \left(U(f) + \frac{\varepsilon}{2} > \mathcal{U}(\mathcal{P}_2, f) \right). \tag{7.4}$$

Let $\mathcal{P}_\varepsilon = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}_\varepsilon$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. By Lemma 7.1 we have

$$\mathcal{L}(\mathcal{P}_\varepsilon, f) \geq \mathcal{L}(\mathcal{P}_1, f),$$
$$\mathcal{U}(\mathcal{P}_\varepsilon, f) \leq \mathcal{U}(\mathcal{P}_2, f),$$

which implies that

$$\mathcal{U}(\mathcal{P}_\varepsilon, f) - \mathcal{L}(\mathcal{P}_\varepsilon, f) \leq \mathcal{U}(\mathcal{P}_2, f) - \mathcal{L}(\mathcal{P}_1, f).$$

Now, $U(f) = L(f)$ by the hypothesis. Hence, using (7.3) and (7.4) we get

$$\mathcal{U}(\mathcal{P}_\varepsilon, f) - \mathcal{L}(\mathcal{P}_\varepsilon, f) \leq U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2} = \varepsilon.$$

$\Longleftarrow$ We assume that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of I such that (7.2) holds. We note that for any partition $\mathcal{P}$ of I we have $\mathcal{L}(\mathcal{P}, f) \leq L(f)$ and $U(f) \leq \mathcal{U}(\mathcal{P}, f)$, which implies that

$$U(f) - L(f) \leq \mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f).$$

Let $\varepsilon > 0$ be given and let $\mathcal{P}_\varepsilon$ be the corresponding partition of I for which (7.2) holds. Thus

$$U(f) - L(f) \leq \mathcal{U}(\mathcal{P}_\varepsilon, f) - \mathcal{L}(\mathcal{P}_\varepsilon, f) < \varepsilon.$$

Since ε is arbitrary, we conclude that $U(f) \leq L(f)$. Hence, by Theorem 7.1

$$U(f) = L(f). \quad \blacksquare$$

Corollary 7.1 *Let f be a bounded function defined on $I = [a, b]$. If there is a sequence $\{\mathcal{P}_n\}$ of partitions of I such that*

$$\lim_{n\to\infty} [\mathcal{U}(\mathcal{P}_n, f) - \mathcal{L}(\mathcal{P}_n, f)] = 0, \tag{7.5}$$

then f is integrable on I and

$$\lim_{n\to\infty} \mathcal{L}(\mathcal{P}_n, f) = \int_a^b f(x)dx = \lim_{n\to\infty} \mathcal{U}(\mathcal{P}_n, f). \tag{7.6}$$

Example 7.4 *Show that $g(x) = x^2$ is integrable on the interval $[0, 1]$.*

Solution. Using the results of Example 7.1 we get

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{L}(\mathcal{P}_n, f) &= \lim_{n\to\infty} \left[\frac{n(n-1)(2n-1)}{6n^3}\right] = \frac{2}{6} = \frac{1}{3}, \\
\lim_{n\to\infty} \mathcal{U}(\mathcal{P}_n, f) &= \lim_{n\to\infty} \left[\frac{n(n+1)(2n+1)}{6n^3}\right] = \frac{2}{6} = \frac{1}{3}.
\end{aligned}$$

Hence, by Corollary (7.1), $g(x) = x^2$ is integrable on the interval $[0, 1]$ and

$$\int_0^1 g(x)dx = \int_0^1 x^2 dx = \frac{1}{3}. \quad \blacksquare$$

We now consider two important classes of Riemann integrable functions.

7.5 Integrability of Monotone Functions

Theorem 7.3 *Any function f that is monotone on $I = [a, b]$ is integrable on I.*

Proof. Suppose that f is nondecreasing on I. Let $\mathcal{P}_n$, $n = 1, 2, \dots$, be the partition of I into n equal subintervals, so that

$$\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}, \quad k = 1, 2, \dots, n.$$

Now $x_{k-1} < x_k \implies f(x_{k-1}) \le f(x_k)$, and we have $m_k = f(x_{k-1}^+)$, $M_k = f(x_k^-)$, $k = 1, 2, \ldots, n$, where

$$f(x^-) = \sup_{t<x} f(t), \quad \text{and} \quad f(x^+) = \inf_{x<s} f(s).$$

Thus

$$\begin{aligned}\sum_{k=1}^{n}(M_k - m_k) &= f(x_1^-) - f(x_o^+) + f(x_2^-) - f(x_1^+) + \cdots + f(x_n^-) - f(x_{n-1}^+)\\ &\le f(x_n) - f(x_o) = f(b) - f(a).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathcal{U}(\mathcal{P}_n, f) - \mathcal{L}(\mathcal{P}_n, f) &= \sum_{k=1}^{n}(M_k - m_k)\Delta x_k = \frac{b-a}{n}\sum_{k=1}^{n}(M_k - m_k)\\ &\le \frac{b-a}{n}(f(b) - f(a)) \to 0, \quad \text{as } n \to 0.\end{aligned}$$

Hence condition (7.5) is satisfied.

The above arguments can be repeated for the case when f is nonincreasing. ∎

Example 7.5 *Let*

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Then f is Riemann integrable on $[0,\ 1]$.

7.6 Integrability of Continuous Functions

Theorem 7.4 *Any function f that is continuous on $I = [a, b]$ is integrable on I.*

Proof. We note that any function that is continuous on a closed interval is uniformly continuous on the interval. Thus f is uniformly continuous on I. This means that

$$\forall \varepsilon_1 > 0\ \exists \delta > 0\ \forall u_1,\ u_2 \in I\ \ (|u_1 - u_2| < \delta \implies |f(u_1) - f(u_2)| < \varepsilon_1). \tag{7.7}$$

Let $\varepsilon > 0$ be given and set $\varepsilon_1 = \varepsilon/(b-a)$. Then (7.7) implies that there is a positive δ such that

$$\forall u_1, u_2 \in I\left(|u_1 - u_2| < \delta \implies |f(u_1) - f(u_2)| < \frac{\varepsilon}{b-a}\right). \tag{7.8}$$

Let $d = (b-a)/\delta$. Choose $\mathcal{P} = \{x_o, x_1, \ldots, x_n\}$ be the partition of I into n equal subintervals, where $n > d$. Then $\Delta_k = x_k - x_{k-1} = (b-a)/n < \delta$, $k = 1, 2, \ldots, n$.

Now, if $u_1,\ u_2 \in [x_{k-1}, x_k]$ then $|u_1 - u_2| \le (b-a)/n < \delta$ and, by (7.8), we have $|f(u_1) - f(u_2)| < \varepsilon/(b-a)$, which implies that $M_k - m_k < (b-a)/n < \delta$. Hence

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Therefore, given $\varepsilon > 0$, we have shown that there is a partition $\mathcal{P}$ of the interval I that satisfies the hypothesis (7.2) of Theorem 7.2. Since ε is arbitrary, by Theorem 7.2, we conclude that f is integrable on I. ∎

Example 7.6 *$f(x) = \sin x$ is Riemann integrable over any interval* $[a, b]$.

7.7 Properties of the Integral

We now establish some basic properties of the Riemann integral, which, in a sense, justify the choice of the word "integral".

Theorem 7.5 *If $f(x)$ is integrable on $[a,b]$, then the function $kf(x)$ is integrable on $[a,b]$ for any constant k, and*

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx. \tag{7.9}$$

Proof.
Case 1: $k > 0$.
To prove that the function $kf(x)$ is integrable on $[a,b]$ requires finding a partition $\mathcal{P}$ of $[a,b]$ such that

$$\mathcal{U}(kf,\mathcal{P}) - \mathcal{L}(kf,\mathcal{P}) < \varepsilon$$

for any $\varepsilon > 0$.
Let ε be given. Since f is integrable on $[a,b]$, there is a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that

$$\mathcal{U}(f,\mathcal{P}_\varepsilon) - \mathcal{L}(f,\mathcal{P}_\varepsilon) < \frac{\varepsilon}{k}.$$

Now

$$\begin{aligned}\mathcal{U}(kf,\mathcal{P}_\varepsilon) - \mathcal{L}(kf,\mathcal{P}_\varepsilon) &= k\ \mathcal{U}(f,\mathcal{P}_\varepsilon) - k\ \mathcal{L}(f,\mathcal{P}_\varepsilon)\\ &= k\ \ (\mathcal{U}(f,\mathcal{P}_\varepsilon) - \mathcal{L}(f,\mathcal{P}_\varepsilon))\\ &< k\cdot\tfrac{\varepsilon}{k} = \varepsilon.\end{aligned}$$

Thus the function $kf(x)$ is integrable on $[a,b]$. It remains to show that (7.9) holds. We have

$$k\mathcal{L}(f,\mathcal{P}_\varepsilon) = \mathcal{L}(kf,\mathcal{P}_\varepsilon) \le \int_a^b kf(x)dx \le \mathcal{U}(kf,\mathcal{P}_\varepsilon) = k\mathcal{U}(f,\mathcal{P}_\varepsilon) \tag{7.10}$$

and

$$\mathcal{L}(f,\mathcal{P}_\varepsilon) \le \int_a^b f(x)dx \le \mathcal{U}(f,\mathcal{P}_\varepsilon)$$

Multiplying the above inequality by k $(k > 0)$ gives

$$k\mathcal{L}(f,\mathcal{P}_\varepsilon) \le k\int_a^b f \le k\mathcal{U}(f,\mathcal{P}_\varepsilon) \tag{7.11}$$

From (7.10) and (7.11) it follows that

$$\forall\varepsilon > 0 \quad \left|\int_a^b kf(x)dx - k\int_a^b f(x)dx\right| \le \varepsilon. \tag{7.12}$$

We conclude that (7.9) holds.
Case 2: $k = 0$.
For every $x \in [a,b]$, $\quad kf(x) = 0$, so that the function kf is integrable by Example 7.3. It is clear that (7.9) holds.
Case 3: $k < 0$. This case is left to the student as an exercise. ∎

Theorem 7.6 *Suppose that f and g are integrable on the interval $I = [a, b]$. Then the function $f + g$ is integrable on I and*

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{7.13}$$

Proof. Let $\mathcal{P}$ be any partition of the interval $I = [a, b]$. Since

$$\inf_{x\in I}[f(x) + g(x)] \geq \inf_{x\in I} f(x) + \inf_{x\in I} g(x),$$

$$\sup_{x\in I}[f(x) + g(x)] \leq \sup_{x\in I} f(x) + \sup_{x\in I} g(x),$$

the lower and upper sums satisfy the inequalities:

$$\mathcal{L}(\mathcal{P}, f + g) \geq \mathcal{L}(\mathcal{P}, f) + \mathcal{L}(\mathcal{P}, g),$$

$$\mathcal{U}(\mathcal{P}, f + g) \leq \mathcal{U}(\mathcal{P}, f) + \mathcal{U}(\mathcal{P}, g).$$

Since the partition $\mathcal{P}$ is arbitrary, we have the following inequalities for the lower and upper integrals of $f + g$, f, and g on the interval I:

$$L(f + g) \geq L(f) + L(g),$$

$$U(f + g) \leq U(f) + U(g).$$

Now for any bounded function, the lower integral is less than or equal to the upper integral, so $L(f + g) \leq U(f + g)$. Thus we have

$$L(f) + L(g) \leq L(f + g) \leq U(f + g) \leq U(f) + U(g).$$

But the functions f and g are integrable on I, so $L(f) = U(f)$, $L(g) = U(g)$, and $L(f) + L(g) = U(f) + U(g)$ implies that

$$L(f + g) = U(f + g).$$

Hence we conclude that the function $f + g$ is integrable on the interval $I = [a, b]$, and (7.13) holds. ∎

Theorem 7.7 *If f is Riemann integrable on $I = [a, b]$, and $f(x) \geq 0$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)dx \geq 0.$$

Proof. Let $\mathcal{P} = \{x_o, x_1, \ldots, x_n\}$ be a partition of I arbitrarily selected from the collection $\boldsymbol{\mathcal{P}}$ of all partitions of I. Then

$$m_k = \inf_{x\in[x_{k-1}, x_k]} f(x) \geq 0, \quad k = 1, 2, \ldots, n,$$

and consequently

$$\mathcal{L}(\mathcal{P}, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \geq 0.$$

Since $\mathcal{L}(\mathcal{P}, f) \geq 0$ for any partition $\mathcal{P} \in \boldsymbol{\mathcal{P}}$, we conclude that

$$L(f) = \sup_{\mathcal{P}\in\boldsymbol{\mathcal{P}}} \mathcal{L}(\mathcal{P}, f) \geq 0.$$

Since f is integrable on $I = [a, b]$, we have

$$\int_a^b f(x)dx = L(f) \geq 0. \quad \blacksquare$$

Theorem 7.8 *If f and g are integrable and bounded on $I = [a, b]$, and if*

$$f(x) \leq g(x), \quad x \in [a, b],$$

then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. By theorem 7.5, the function $-f$ is integrable, so is $g - f = g + (-f)$, and

$$\int_a^b [g(x) - f(x)]dx = \int_a^b g(x)dx - \int_a^b f(x)dx.$$

By the hypothesis, $g(x)-f(x) \geq 0$. Applying Theorem 7.7 to the function $g-f$, we conclude that

$$\int_a^b [g(x) - f(x)]dx = \int_a^b g(x)dx - \int_a^b f(x)dx \geq 0, \quad \text{or} \quad \int_a^b g(x)dx \geq \int_a^b f(x)dx. \quad \blacksquare$$

Theorem 7.9 *If f is integrable on $I = [a, b]$ and*

$$m \leq f(x) \leq M, \quad x \in I,$$

then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof. By Example 7.3, the constant functions, $g_1(x) = m$ and $g_2(x) = M$ are integrable on I and

$$\int_a^b g_1(x)dx = \int_a^b mdx = m(b-a), \quad \int_a^b g_2(x)dx = \int_a^b Mdx = M(b-a).$$

Hence, by Theorem 7.8,

$$\begin{aligned} &m \leq f(x) \leq M, \quad x \in I = [a, b] \quad \Longrightarrow \\ &m(b-a) = \int_a^b mdx \leq \int_a^b f(x)dx \leq \int_a^b Mdx = M(b-a), \end{aligned}$$

and the proof is complete. $\blacksquare$

Theorem 7.10 *Let f be a bounded function on the interval $I = [a, b]$ and let c be any interior point of I that splits I up into $I_1 = [a, c]$ and $I_2 = [c, d]$.*

The function f is integrable on I if and only if f is integrable on both I_1 and I_2. In this case

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx. \tag{7.14}$$

The proof of this theorem makes use of the following lemma.

Lemma 7.3 *Suppose that the assumptions of Theorem **7.10** hold. Let $L(f)$, $L_1(f)$, and $L_2(f)$ denote the lower integrals of f on the intervals I, I_1, and I_2, respectively. Similarly, let $U(f)$, $U_1(f)$, and $U_2(f)$ denote the upper integrals of f on I, I_1, and I_2, respectively. Then we have*

(i) $L(f) = L_1(f) + L_2(f)$,

(ii) $U(f) = U_1(f) + U_2(f)$.

Proof of Lemma.

We shall only prove the **(ii)** part of the lemma. The proof of **(i)** proceeds in a similar way.

Let $\mathcal{P}$ be any partition of $[a, b]$. Then $\mathcal{P}_1 = (\mathcal{P} \cup \{c\}) \cap [a, c]$ is a partition of $[a, c]$ and $\mathcal{P}_2 = (\mathcal{P} \cup \{c\}) \cap [c, d]$ is a partition of $[c, b]$. From the definition of the upper sums, we have

$$\mathcal{U}(\mathcal{P} \cup \{c\}, f) = \mathcal{U}(\mathcal{P}_1, f) + \mathcal{U}(\mathcal{P}_2, f).$$

Also, since $\mathcal{P} \cup \{c\}$ is a refinement of $\mathcal{P}$, $\mathcal{U}(\mathcal{P} \cup \{c\}, f) \leq \mathcal{U}(\mathcal{P}, f)$. Hence

$$\begin{aligned} \mathcal{U}(\mathcal{P}, f) &\geq \mathcal{U}(\mathcal{P}_1, f) + \mathcal{U}(\mathcal{P}_2, f) \\ &\geq U_1(f) + U_2(f). \end{aligned}$$

Since this inequality holds for all partitions $\mathcal{P}$ of I, we conclude that

$$U(f) \geq U_1(f) + U_2(f).$$

For the reverse inequality, let $\varepsilon > 0$ be given. By definition of the upper integral, there is a partition $\mathcal{P}_1$ of $[a, c]$ and a partition $\mathcal{P}_2$ of $[c, b]$ for which

$$U_1(f) \leq \mathcal{U}(\mathcal{P}_1, f) < U_1(f) + \frac{\varepsilon}{2}$$

$$U_2(f) \leq \mathcal{U}(\mathcal{P}_2, f) < U_2(f) + \frac{\varepsilon}{2}.$$

Then

$$\mathcal{U}(\mathcal{P}_1 \cup \mathcal{P}_2, f) = \mathcal{U}(\mathcal{P}_1, f) + \mathcal{U}(\mathcal{P}_2, f) \leq U_1(f) + U_2(f) + \varepsilon.$$

But $U(f) \leq \mathcal{U}(\mathcal{P}_1 \cup \mathcal{P}_2, f)$, so that $U(f) < U_1(f) + U_2(f) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$U(f) \leq U_1(f) + U_2(f). \quad \blacksquare$$

Proof of Theorem 7.10.

$\Longrightarrow$ We assume that f is integrable on I, so that $L(f) = U(f)$ and, using the statements **(i)** and **(ii)** of Lemma **7.3**, we conclude that

$$L_1(f) + L_2(f) = L(f) = U(f) = U_1(f) + U_2(f). \tag{7.15}$$

Now, by Theorem 7.1, $L_1(f) \leq U_1(f)$ and $L_2(f) \leq U_2(f)$. We claim that $L_1(f) = U_1(f)$ and $L_2(f) = U_2(f)$, for if either $L_1(f) < U_1(f)$ or $L_2(f) < U_2(f)$, we would have a contradiction to (7.15). Hence f is integrable on I_1 and on I_2 and (7.14) holds.

$\Longleftarrow$ We assume that f is integrable on I_1 and on I_2, so that $L_1(f) = U_1(f)$ and $L_2(f) = U_2(f)$. Now, by **(i)** and **(ii)** of Lemma **7.3**, we get

$$U(f) = U_1(f) + U_2(f) = L_1(f) + L_2(f) = L(f),$$

so that $L(f) = U(f)$, meaning that f is integrable on I and (7.14) holds. ∎

7.8 Integrability of Composite Functions

Theorem 7.11 The Composite Theorem

Let $I = [a, b]$ and $J = [c, d]$ be intervals and let $\varphi : J \mapsto \mathbb{R}$ be continuous and assume that $f : I \mapsto J$ is Riemann integrable. Then the composite function $\varphi \circ f : I \mapsto \mathbb{R}$ is Riemann integrable on I.

Proof. We shall show that for every $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of I such that

$$\mathcal{U}(\mathcal{P}_\varepsilon, \varphi \circ f) - \mathcal{L}(\mathcal{P}_\varepsilon, \varphi \circ f) < \varepsilon, \tag{7.16}$$

so that, by Theorem 7.2, the function $\varphi \circ f$ is integrable on I.

We note that the function φ, being continuous on a closed interval is uniformly continuous on this interval. Thus φ is uniformly continuous on $J = [c, d]$ and we have

$$\forall \varepsilon_1 > 0 \ \exists \delta > 0 \quad (s,\ t \in J \ \& \ |s - t| < \delta \quad \Longrightarrow \quad |\varphi(s) - \varphi(t)| < \varepsilon_1). \tag{7.17}$$

Since f is integrable on I, for every $\varepsilon_2 > 0$, there exists a partition $\mathcal{P}$ of the interval I, such that

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \varepsilon_2.$$

We consider an arbitrary value of ε_1 and the corresponding value of δ; and an arbitrary value of ε_2 and the corresponding partition $\mathcal{P} = \{x_o, x_1, \ldots, x_n\}$ of the interval I.

The partition $\mathcal{P}$ divides I into n subintervals $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$. Let

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x).$$

We shall consider separately those subintervals $[x_{k-1}, x_k]$ for which $M_k - m_k < \delta$. Let A denote the corresponding set of indices:

$$A = \{k : \quad M_k - m_k < \delta\},$$

and let

$$B = \{k : \quad M_k - m_k \geq \delta\}.$$

If we denote by $\tilde{m}_k$ and $\tilde{M}_k$ the infimum and supremum of the function $\varphi \circ f$ on the interval $[x_{k-1}, x_k]$, then we can write

$$\tilde{M}_k - \tilde{m}_k = \sup_{x,y \in [x_{k-1}, x_k]} (\varphi \circ f(x) - \varphi \circ f(y)).$$

Now, if $k \in A$ and $x,\ y \in [x_{k-1}, x_k]$, then $|f(x) - f(y)| < \delta$, which, by (7.17), implies that $|\varphi \circ f(x) - \varphi \circ f(y)| < \varepsilon_1$, so that $\tilde{M}_k - \tilde{m}_k \leq \varepsilon_1$ for $k \in A$. We conclude, therefore, that

$$\sum_{k \in A} (\tilde{M}_k - \tilde{m}_k)(x_k - x_{k-1}) \leq \varepsilon_1 (b - a). \tag{7.18}$$

On the other hand, if $k \in B$, we have $\tilde{M}_k - \tilde{m}_k \leq 2K$, where $K = \sup_{t \in J} |\varphi(t)|$. Hence

$$\sum_{k \in B} (\tilde{M}_k - \tilde{m}_k)(x_k - x_{k-1}) \leq 2K \sum_{k \in B} (x_k - x_{k-1}).$$

If $k \in B$ then $M_k - m_k \geq \delta \implies \frac{1}{\delta}(M_k - m_k) \geq 1$ and we can write

$$\begin{aligned} \sum_{k \in B} (x_k - x_{k-1}) &\leq \frac{1}{\delta} \sum_{k \in B} (M_k - m_k)(x_k - x_{k-1}) \\ &\leq \frac{1}{\delta} \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{1}{\delta} (\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f)) < \frac{1}{\delta} \varepsilon_2. \end{aligned}$$

Hence

$$\sum_{k \in B} (\tilde{M}_k - \tilde{m}_k)(x_k - x_{k-1}) \leq \frac{2K}{\delta} \varepsilon_2. \tag{7.19}$$

Combining (7.18) and (7.19) gives

$$\mathcal{U}(\mathcal{P}, \varphi \circ f) - \mathcal{L}(\mathcal{P}, \varphi \circ f) \leq D = \varepsilon_1 (b - a) + \frac{2K}{\delta} \varepsilon_2. \tag{7.20}$$

The above holds for any $\varepsilon_1 > 0$ and any $\varepsilon_2 > 0$, where δ corresponds to ε_1 as specified in (7.17). Without loss of generality, we select $\delta < \varepsilon_1$.

The objective of the proof is to show that (7.16) holds for any $\varepsilon > 0$.

Let $\varepsilon > 0$ be given. If we set

$$\varepsilon_1 = \frac{\varepsilon}{b - a + 2K}, \quad \varepsilon_2 = \delta^2,$$

and recall that $\delta < \varepsilon_1$, we get D of (7.20) equal to ε:

$$D = \varepsilon_1(b-a) + 2K\delta < \varepsilon_1(b-a) + 2K\varepsilon_1 = \varepsilon_1(b-a+2K) = \varepsilon.$$

Hence we have shown that there exists a partition $\mathcal{P}_\varepsilon = \mathcal{P}$ of the interval I such that

$$\mathcal{U}(\mathcal{P}, \varphi \circ f) - \mathcal{L}(\mathcal{P}, \varphi \circ f) \leq \varepsilon,$$

which means that the function $\varphi \circ f$ is integrable on the interval I. ∎

7.9 Further Properties of the Integral

Theorem 7.12 *If f is integrable on $I = [a, b]$ then the function $|f|$ is integrable on I, and*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Proof. Since f is integrable on I, f is bounded, so there exists $K > 0$ such that $|f(x)| \leq K, \ \ x \in I$.

Let $J = [-K, K]$. Define $\varphi : J \mapsto \mathbb{R}$ as $\varphi(t) = |t|$, and note that the composite function $\varphi \circ f$ defined on I is $\varphi \circ f = |f|$.

Since $\varphi(t) = |t|$ is continuous on J, the Composite Theorem applies to conclude that

$$\varphi \circ f = |f|$$

is integrable on $I = [a, b]$.

To prove the inequality, note that $f(x) \leq |f(x)|$, and $-f(x) \leq |f(x)|$, so $-|f(x)| \leq f(x) \leq |f(x)|$. Hence, by Theorem (7.8), we have

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx. \ \blacksquare$$

Theorem 7.13 *Let f be integrable on $I = [a, b]$. Then the function f^n, for any natural number n, is integrable on I.*

Proof. Since f is integrable, f is bounded on I, so that $|f(x)| \leq K, \ \ x \in I$, for some $K > 0$. Let $\varphi(t) = t^n$ for $t \in J = [-K, K]$. Then $\varphi \circ f = f^n$ and the Composite Theorem applies. ∎

Theorem 7.14 The Product theorem
If f and g are both integrable on $I = [a, b]$, then so is the product $f \cdot g$.

Proof. We have

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]. \ \blacksquare$$

Theorem 7.15 *Suppose that f is integrable on $I = [a,b]$ and $f(x) \geq \delta, \ \ x \in I$ for some $\delta > 0$. Then the function $\dfrac{1}{f}$ is integrable on I.*

Proof. Since f is integrable on I, f is bounded on I. Thus we have

$$\delta \leq f(x) \leq K, \ \ x \in I.$$

Let $J = [\delta, K]$ and define $\varphi(t)$ on J as $\varphi(t) = \dfrac{1}{t}$. Clearly, φ is continuous on J and $\varphi \circ f = \frac{1}{f}$ is integrable on I by the Composite Theorem. ∎

Theorem 7.16 Schwarz's Inequality.
If f and g are integrable over the interval $I = [a,b]$, then

$$\left(\int_a^b f(x) \cdot g(x) dx\right)^2 \leq \int_a^b [f(x)]^2 dx \cdot \int_a^b [g(x)]^2 dx. \tag{7.21}$$

Proof. Let λ be a constant and consider the expression $(f + \lambda g)^2$ which is nonnegative. We have

$$(f + \lambda g)^2 = f^2 + 2\lambda fg + \lambda^2 g^2 \geq 0,$$

for all real values of the constant λ.

Since both f and g are integrable then, we conclude that

$$\lambda^2 \int_a^b g^2 dx + 2\lambda \int_a^b fg dx + \int_a^b f^2 dx \geq 0. \tag{7.22}$$

Now the quadratic in λ, $A\lambda^2 + B\lambda + C$, where $A > 0$, is nonnegative for all $\lambda \in \mathbb{R}$ if and only if $\Delta = B^2 - 4AC \leq 0$. Hence (7.22) holds if and only if

$$4\left(\int_a^b f(x) \cdot g(x) dx\right)^2 - 4\left(\int_a^b g^2(x) dx\right) \cdot \left(\int_a^b f^2(x) dx\right) \leq 0,$$

which proves the Schwarz's inequality (7.21). ∎

7.10 The Fundamental Theorem of Integral Calculus

Theorem 7.17 *Suppose that f is integrable over the interval $I = [a,b]$ and let F be defined on I by*

$$F(x) = \int_a^x f(t) dt, \quad x \in I. \tag{7.23}$$

(a) *F is a continuous function of x in $I = [a,b]$.*

(b) *F is differentiable at any point $c \in I$ at which f is continuous, and*

$$F'(c) = f(c).$$

Proof.
(a) Since f is integrable on I, f is bounded on I, so that

$$|f(t)| \le M = \sup_{x\in I} |f(x)|, \quad t \in I.$$

Let $x_1,\ x_2 \in [a,b],\ \ x_1 < x_2$. We have

$$\begin{aligned} |F(x_1) - F(x_2)| &= \left|\int_a^{x_1} f(t)dt - \int_a^{x_2} f(t)dt\right| = \left|\int_{x_1}^{x_2} f(t)dt\right| \\ &\le \int_{x_1}^{x_2} |f(t)|dt \le \int_{x_1}^{x_2} M dt = |x_1 - x_2|M. \end{aligned}$$

We can see, therefore, that

$$\forall\varepsilon > 0\ \exists\delta = \frac{\varepsilon}{M} > 0 \quad ((x_1,\ x_2 \in I,\ \ |x_1 - x_2| < \delta) \quad \Longrightarrow \quad |F(x_1) - F(x_2)| < \varepsilon),$$

which means that the function $F(x)$ defined by (7.23) is continuous in the interval $I = [a,b]$.
(b) Since f is continuous at the point c, we have

$$\forall\varepsilon > 0\ \exists\delta > 0 \quad (|t - c| < \delta \quad \Longrightarrow \quad |f(t) - f(c)| < \varepsilon). \tag{7.24}$$

We can write

$$\frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt,$$

since

$$\begin{aligned} F(c+h) - F(c) &= \int_c^{c+h} f(t)dt, \\ f(c) &= \frac{1}{h}\int_c^{c+h} f(c)dt. \end{aligned}$$

Let $\varepsilon > 0$ be given and let $\delta(\varepsilon)$ be the corresponding value of δ in (7.24). If $|h| < \delta(\varepsilon)$ then, on application of Theorems 7.12 and 7.9, we get

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt\right| \le \frac{1}{|h|}\int_c^{c+h} |f(t) - f(c)|dt \le \frac{1}{|h|}\varepsilon|h| = \varepsilon.$$

Since ε is arbitrary, we conclude that

$$\lim_{h\to 0}\left(\frac{F(c+h) - F(c)}{h} - f(c)\right) = 0.$$

Hence

$$F'(c) = \lim_{h\to 0}\frac{F(c+h) - F(c)}{h}$$

exists and equals to $f(c)$: $\quad F'(c) = f(c)$. ∎

Comments: The above theorem ensures that every continuous function possesses an antiderivative. It also gives a method for evaluating $\int_{x_1}^{x_2} f(t)dt$, provided that an antiderivative of f is known. If F is an antiderivative of f, then

$$\int_{x_1}^{x_2} f(t)dt = \int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt = F(t)\Big|_{x_1}^{x_2} = F(x_2) - F(x_1).$$

We recall that two antiderivatives of f differ by a constant, so that F may be chosen to be any antiderivative of f. In particular, when $f(x) = h'(x)$, we obtain

$$\int_a^b h'(x)dx = h(b) - h(a). \tag{7.25}$$

It is clear that the definition of the Riemann integral is closely related to the geometric notion of area. The properties of the integral were developed independently of this and, in fact, one may define the area A under $y = f(x),\ \ a \le x \le b$, to be

$$A = \int_a^b |f(x)|dx,$$

as shown in Figure 7.2 A simple example involving the calculation of the "area under a curve" is given in Section 7.13 (see Example 7.9).

(a) $y = f(x)$ **(b)** $y = |f(x)|$

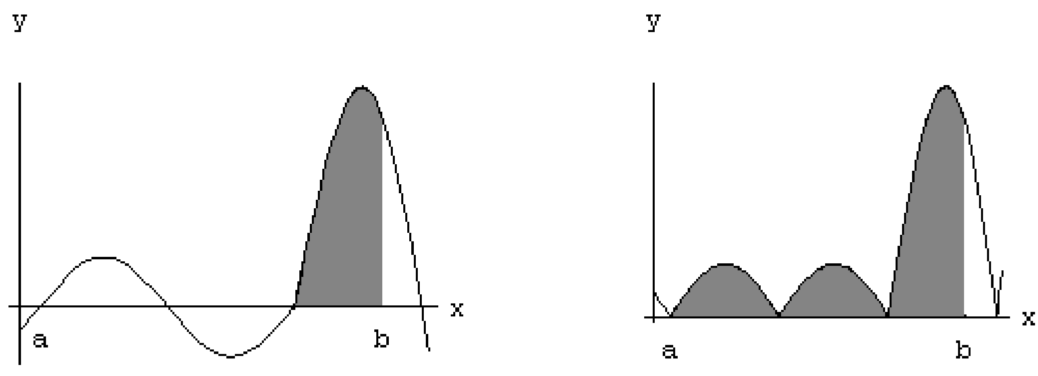


Figure 7.2: The Area, $\int_a^b f(x)dx$, and $\int_a^b |f(x)|dx$.

7.11 Integration by Parts

Theorem 7.18 Integration by Parts *Consider two functions f and g defined on a closed interval $I = [a, b]$. If f and g have continuous derivatives on I then*

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx. \tag{7.26}$$

Proof. Let h denote the product of the functions f and g:

$$h(x) = f(x)g(x), \quad x \in I.$$

We note that the function h is differentiable and its derivative

$$h'(x) = f'(x)g(x) + f(x)g'(x), \quad x \in I,$$

is a continuous function on $I = [a, b]$. Hence

$$\begin{aligned} \int_a^b (f'(x)g(x) + f(x)g'(x))dx &= \int_a^b h'(x)dx = h(b) - h(a) \\ &= f(b)g(b) - f(a)g(a). \end{aligned}$$

Hence

$$\int_a^b (f'(x)g(x) + f(x)g'(x))dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b,$$

which implies (7.26). ∎

Theorem 7.19 (The Mean Value Theorem for Integrals)
Let f and g be continuous functions on the closed interval $[a, b]$ and suppose that $g(x) \geq 0$ for $x \in [a, b]$. Then there is some value c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx. \tag{7.27}$$

Proof. Since f is continuous on a closed interval, f is bounded. Let

$$m = \min_{x \in [a,b]} f(x) \quad \text{and} \quad M = \max_{x \in [a,b]} f(x),$$

so that

$$m \leq f(x) \leq M, \quad x \in [a, b].$$

Since $g(x) \geq 0, \quad x \in [a, b]$, we have

$$mg(x) \leq f(x)g(x) \leq Mg(x), \quad x \in [a, b].$$

Hence, by theorem 7.8,

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx. \tag{7.28}$$

If $\int_a^b g(x)dx = 0$, then (7.27) holds for every choice of c. So, we assume $\int_a^b g(x)dx \neq 0$. Then (7.27) implies that

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M. \tag{7.29}$$

Now, the function f as a continuous function on the interval $[a, b]$, must assume every value between m and M. In particular, there is a point c in the interval $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

and (7.27) holds for this value of c. ∎

Comments: The theorem is also valid when $g(x) \le 0$ for all $x \in [a, b]$ and the proof is similar. In this case, $\displaystyle\int_a^b g(x)dx \le 0$, and the inequality (7.29) still holds, when the denominator is not zero.

Suppose that the function g is the constant function, $f(x) \equiv 1$ for all $x \in [a, b]$. Then we obtain, as a special case, the following Corollary.

Corollary 7.2 *If f is continuous on $[a, b]$, then there is some value c in (a, b) such that*

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c).$$

Theorem 7.20 The second Mean Value Theorem for Integrals
Let f, f', and g be continuous on $I = [a, b]$, with $f' \ge 0$ in I. Then there exists a number ξ, $a < \xi < b$, such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx.$$

Proof. Let $G(x) = \displaystyle\int_a^x g(t)dt$. Integrating by parts, we get

$$\int_a^b f(x)g(x)dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)dx = f(b)G(b) - \int_a^b f'(x)G(x)dx,$$

since $G(a) = 0$. By the Mean Value Theorem for integrals, we conclude that there is a $\xi \in I$, such that

$$\int_a^b f'(x)G(x)dx = G(\xi)\int_a^b f'(x)dx.$$

Hence

$$\int_a^b f'(x)G(x)dx = G(\xi)[f(b) - f(a)] = f(b)G(\xi) - f(a)G(\xi), \quad a < \xi < b.$$

Therefore

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= f(b)G(b) - f(b)G(\xi) + f(a)G(\xi) \\
&= f(b)\left[\int_a^b g(t)dt - \int_a^\xi g(t)dt\right] + f(a)\int_a^\xi g(t)dt \\
&= f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx, \quad a < \xi < b.
\end{aligned}$$ ∎

7.12 Taylor's Theorem Revisited

Theorem 7.21 Taylor's Theorem with integral form of the Remainder. *Let f be $(n+1)$ times differentiable on the interval $|x-x_o|<h,\ \ h>0$. Then for all x in this interval we have*

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_o)}{k!}(x-x_o)^k+R_n(x), \tag{7.30}$$

where

$$R_n(x)=\int_{x_o}^{x}\frac{f^{(n+1)}(t)}{n!}(x-t)^n dt. \tag{7.31}$$

Proof. (by Mathematical Induction)
Step 1. If $n=0$ then (7.30) reduces to

$$f(x)=f(x_o)+R_o(x),$$

where $R_o(x)$, as defined by (7.31), is

$$R_o(x)=\int_{x_o}^{x}f'(t)dt.$$

Thus, using (7.25), we have

$$R_o(x)=f(x)-f(x_o)$$

and it is clear that (7.30) is valid for $n=0$.
Step 2. Assume that (7.30) and (7.31) are valid for a particular value of n.
Now integrate (7.31) by parts to obtain

$$\begin{aligned}R_n(x) &= \frac{1}{n!}\int_{x_o}^{x}f^{(n+1)}(t)(x-t)^n dt\\ &= -f^{(n+1)}(t)\left.\frac{(x-t)^{n+1}}{n!(n+1)}\right|_{x_o}^{x}+\frac{1}{n!(n+1)}\int_{x_o}^{x}f^{(n+2)}(t)(x-t)^{n+1}dt\\ &= \frac{1}{(n+1)!}f^{(n+1)}(x_o)(x-x_o)^{n+1}+R_{n+1}(x).\end{aligned}$$

Inserting the above expression for $R_n(x)$ into (7.30) gives

$$f(x)=\sum_{k=0}^{n+1}\frac{f^{(k)}(x_o)}{k!}(x-x_o)^k+R_{n+1}(x),$$

and proves the theorem for the case $n+1$. The proof is complete. ∎

Example 7.7 *Suppose that f is $(n+1)$-times continuously differentiable on a given interval $[x_o,x]$. Apply the Mean Value Theorem (for integrals) to prove that*

$$R_n(x)=\frac{f^{(n+1)}(\xi)(x-x_o)^{n+1}}{(n+1)!}=\int_{x_o}^{x}\frac{f^{(n+1)}(t)}{n!}(x-t)^n dt, \tag{7.32}$$

for some value of ξ in the interval (x_o,x).

Solution. Replace x by t in Theorem 7.19 and apply it to the interval $[x_o, x]$ to get

$$\int_{x_o}^{x} f(t)g(t)dt = f(\xi)\int_{x_o}^{x} g(t)dt \quad \text{for some } \xi \in (x_o, x).$$

Replace $f(x)$ by $f^{(n+1)}(t)$ and set $g(t) = (x-t)^n$, $t \in [x_o, x]$, so $g(t) \geq 0$ for $t \in [x_o, x]$. We obtain

$$\begin{aligned}\int_{x_o}^{x} f^{(n+1)}(t)(x-t)^n dt &= f^{(n+1)}(\xi)\int_{x_o}^{x}(x-t)^n dt \\ &= f^{(n+1)}(\xi)\frac{(x-x_o)^{n+1}}{n+1},\end{aligned}$$

for some value of ξ in $[x_o, x]$. ∎

Example 7.8

In Example 6.3, we found the Taylor polynomial of degree $= 2k+1$ for the function $f(x) = \sin x$:

$$P_{2k+1,\,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Find the integral form of the remainder

$$R_{2k+1}(x) = f(x) - P_{2k+1,\,0}(x).$$

Compute $\sin 2$ with an error of less than 10^{-4}.

Solution. The remainder is given by

$$R_{2k+1}(x) = \int_0^x \frac{f^{(2k+2)}(t)}{(2k+1)!}(x-t)^{2k+1}dt.$$

Since $|f^{(k)}(x)| \leq 1$, $k = 1, 2, \ldots$, we get

$$|R_{2k+1}(x)| \leq \frac{1}{(2k+1)!}\left|\int_0^x (x-t)^{2k+1}dt\right|,$$

and

$$\int_0^x (x-t)^{2k+1}dt = -\left.\frac{(x-t)^{2k+2}}{2k+2}\right|_{t=0}^{t=x} = \frac{x^{2k+2}}{2k+2}.$$

Thus we obtain the following estimate of the remainder $R_{2k+1}(x)$:

$$|R_{2k+1}(x)| \leq \frac{|x|^{2k+1}}{(2k+2)!}.$$

Therefore,

$$\sin 2 = P_{2k+1}(2) + R_{2k+1}(2),$$

where

$$|R_{2k+1}| \le \frac{2^{2k+2}}{(2k+2)!}.$$

Hence, $P_{2k+1}(2)$ approximates $\sin 2$ with an error of less than 10^{-4} provided that

$$\frac{2^{2k+2}}{(2k+2)!} < 10^{-4}. \tag{7.33}$$

By a straightforward substitution of $k = 1, 2, 3, 4$, and 5 into the expression $\dfrac{2^{2k+2}}{(2k+2)!}$ we can see that (7.33) holds for $k \ge 5$.

Therefore, $\sin 2$ can be computed as

$$\sin 2 \approx 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \frac{2^{11}}{11!}$$

with an error of less than 10^{-4}. ∎

7.13 Integration by Substitution

Theorem 7.22 First Substitution Theorem. *Consider a function φ with domain $J = [\alpha, \beta]$ and range $I = \varphi(J)$ and consider a function $f(x)$ defined on I:*

$$\varphi : \; J = [\alpha, \beta] \;\mapsto\; I, \quad f : \; I \mapsto \; \mathbb{R}.$$

If f is continuous on I and φ has a continuous derivative on J, then

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_a^b f(u)du, \tag{7.34}$$

where $u = \varphi(t)$, $a = \varphi(\alpha)$ and $b = \varphi(\beta)$.

Proof. Since f is continuous on I, we can define a function F on I by

$$F(u) = \int_a^u f(x)dx, \;\; u \in I.$$

We have

$$F(a) = 0, \quad F(b) = \int_a^b f(x)dx \tag{7.35}$$

and

$$F'(u) = f(u), \;\; u \in I.$$

Let

$$H(t) = F(\varphi(t)), \;\; t \in J.$$

By the Chain Rule,

$$H'(t) = f(\varphi(t))\varphi'(t), \;\; t \in J.$$

Since $H(\alpha) = F(\varphi(\alpha)) = F(a) = 0$, we have

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_\alpha^\beta H'(t)dt = H(\beta) - H(\alpha) = H(\beta).$$

On the other hand, using (7.35), we have

$$H(\beta) = F(\varphi(\beta)) = F(b) = \int_a^b f(x)dx.$$

The last two equations prove (7.34). ∎

Example 7.9 *Show that the area A of the ellipse defined by the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.36}$$

is $A = \pi ab$.

Solution. The area enclosed by the ellipse (7.36) is, by symmetry, four times the area in the first quadrant (see the diagram).

Thus,

$$A = 4\int_0^a ydx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx.$$

Put $x = a\sin\theta$, to get

$$\begin{aligned} A &= \frac{4b}{a}\int_0^{\pi/2}(a\cdot\cos\theta)\cdot a\cos\theta d\theta \\ &= 4ab\int_0^{\pi/2}\cos^2\theta d\theta \\ &= 2ab\int_0^{\pi/2}(\cos 2\theta + 1)d\theta \\ &= 2ab\cdot\left[\frac{\sin 2\theta}{2}\bigg|_0^{\pi/2} + \theta\bigg|_0^{\pi/2}\right] \\ &= 2ab\cdot\frac{\pi}{2} \\ &= \pi ab. \end{aligned}$$

Note. When $a = b$, the ellipse (7.36) becomes a circle of radius a. The above formula reduces to a familiar one: $A = \pi a^2$. ∎

7.14 Exercises

7.1 Use the definition of the integral to prove that

$$\int_a^b xdx = \frac{1}{2}(b^2 - a^2).$$

7.2 Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational.} \end{cases}$$

Evaluate, for a given partition $\mathcal{P}$, the upper and lower sums $\mathcal{U}(\mathcal{P}, f)$, $\mathcal{L}(\mathcal{P}, f)$. Hence determine the upper and lower integrals $U(f)$, $L(f)$ and decide whether or not the function is Riemann integrable.

7.3 Let

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Find a formula for $F(x) = \int_0^x f(t)dt$.

Is it true that $F'(x) = f(x)$?

Compare with **Comments** to Theorem 7.17.

7.4 Assume g, f are continuous functions on $[a,b]$ with f having a continuous non-negative derivative in the same interval.

(i) Show that there is c, $a < c < b$, for which

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Hint: Let $G(x) = \int_a^x g(t)dt$, so that $G' = g$.

(ii) Show that there is d, $a < d < b$, for which

$$\int_a^b f(x)g(x) = f(a)\int_a^d g(x)dx.$$

7.5 Show that $\left|\int_x^{x'} \frac{\sin x}{x} dx\right| \le \frac{2}{x}$ whenever $x' > x > 0$.

Hint: Use 7.4 (i) above.

7.6 If f'' exists on (a,b), show that for x and h with x, $x+h$, $x-h$ in (a,b), there is c between $x+h$ and $x-h$ such that

$$f(x+h) + f(x-h) - 2f(x) = h^2 f''(c).$$

7.7 Show that

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)^3 f''(c),$$

where $a < c < b$.

Chapter 8

Improper Integrals and Applications

8.1 Improper Integrals

The Definition of Riemann integral $\int_a^b f(x)dx$ is only meaningful when the integrand $f(x)$ is a bounded function defined on the closed interval $[a, b]$.

In this section we shall extend the definition of the integral $\int_a^b f(x)dx$ to the situation where the interval of integration is infinite and $f(x)$ is not necessarily bounded over the interval (a, b). We shall consider integrals

$$\int_{-\infty}^{b} f(x)dx, \quad \int_{a}^{+\infty} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx,$$

when one or both limits of integration are infinite, as well as $\int_a^b f(x)dx$, when $f(x)$ has infinite singularity at one or more points of the interval $[a, b]$.

Example 8.1

(a) Consider the function $f(x) = \dfrac{1}{1+x^2}$ and refer to Figure 8.1(a) that shows the area $A(\lambda)$ under the curve $y = \dfrac{1}{1+x^2}$ over the interval $[0, \lambda]$, where $0 < \lambda < 10$. If $\lambda \to +\infty$, then the limit $\lim\limits_{\lambda \to +\infty} A(\lambda)$ can be considered as the area under the curve $y = \dfrac{1}{1+x^2}$ over the interval $[0, +\infty)$. Hence we define

$$\int_0^{+\infty} \frac{1}{1+x^2}dx$$

to stand for the limit $\lim\limits_{\lambda \to +\infty} A(\lambda)$, provided that this limit exists:

$$\int_0^{+\infty} \frac{1}{1+x^2}dx = \lim_{\lambda \to +\infty} A(\lambda) = \lim_{\lambda \to +\infty} \int_0^{\lambda} \frac{1}{1+x^2}dx.$$

In Example 8.2(b) we shall show that $A(\lambda) \to \frac{\pi}{2}$, as $\lambda \to +\infty$.

(b) Consider the function $f(x) = \dfrac{1}{\sqrt{x}}$, that is defined for all $x \neq 0$. Refer to Figure 8.1(b) to see the area $B(\varepsilon)$ under the curve $y = \frac{1}{\sqrt{x}}$ over the interval $[\varepsilon, 1]$, where $0 < \varepsilon < 1$. It is reasonable to define

$$\int_0^1 \frac{1}{\sqrt{x}} dx$$

to stand for the limit $\lim\limits_{\varepsilon \to 0+} B(\varepsilon)$, provided that this limit exists. We shall show in Example 8.4(a) that $B(\varepsilon) \to 2$, as $\varepsilon \to 0+$. ∎

(a) $A(\lambda) = \displaystyle\int_0^\lambda \frac{1}{1+x^2} dx$, $\lambda = 8, 9$.

(b) $B(\varepsilon) = \displaystyle\int_\varepsilon^1 \frac{1}{\sqrt{x}} dx$, $\varepsilon = .01,\ .05$.

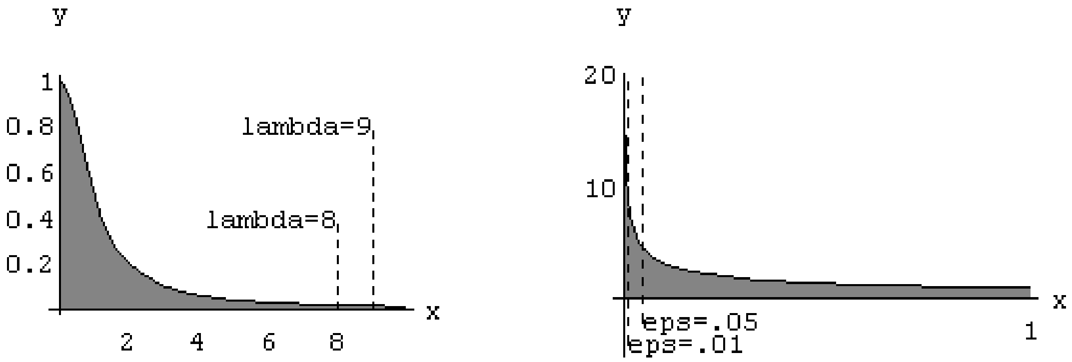


Figure 8.1: Illustrating convergence of $A(\lambda) = \displaystyle\int_0^\lambda f(x)dx$, as $\lambda \to \infty$, and convergence of $B(\varepsilon) = \displaystyle\int_\varepsilon^1 f(x)dx$, as $\varepsilon \to 0+$.

8.2 Improper Integrals of the First Kind

In this section we examine the effect of relaxing the condition that the interval of integration is a closed, bounded interval $[a, b]$ in the definition of Riemann integral. Thus, we replace one or both of a and b in $\int_a^b f(x)dx$ by $\pm\infty$ and introduce the symbols

$$\int_{-\infty}^b f(x)dx, \quad \int_a^{+\infty} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx,$$

(called infinite integrals of $f(x)$) to stand for the appropriate limits as defined below.

We define the infinite integral of $f(x)$ over the interval $[a, +\infty]$ as the limit of $\int_a^\lambda f(x)dx$, when $\lambda \to +\infty$, provided that this limit exists.

We define the infinite integral of $f(x)$ over the interval $(-\infty, b]$ as the limit of $\int_{-\lambda}^b f(x)dx$, when $\lambda \to +\infty$, provided that this limit exists.

Definition 8.1 *Let f be bounded and integrable over the interval $[a, \lambda]$ for every $\lambda > a$. If*

$$\lim_{\lambda\to\infty}\int_a^{\lambda} f(x)dx \tag{8.1}$$

exists, then we say that $\int_a^{\infty} f(x)dx$ **converges** *and write*

$$\int_a^{\infty} f(x)dx = \lim_{\lambda\to\infty}\int_a^{\lambda} f(x)dx.$$

If the limit (8.1) does not exist, we say that $\int_a^{\infty} f(x)dx$ **diverges**.

If the limit (8.1) is infinite, we say that $\int_a^{\infty} f(x)dx$ **diverges to** $-\infty/+\infty$ *and write*

$$\int_a^{\infty} f(x)dx = -\infty, \quad \text{or} \quad \int_a^{\infty} f(x)dx = +\infty,$$

respectively.

Definition 8.2 *Let f be bounded and integrable over the interval $[-\lambda, b]$ for every λ, such that $-\lambda < b$. If*

$$\lim_{\lambda\to\infty}\int_{-\lambda}^{b} f(x)dx \tag{8.2}$$

exists, then we say that $\int_{-\infty}^{b} f(x)dx$ **converges** *and write*

$$\int_{-\infty}^{b} f(x)dx = \lim_{\lambda\to\infty}\int_{-\lambda}^{b} f(x)dx.$$

If the limit (8.2) does not exist, we say that $\int_{-\infty}^{b} f(x)dx$ **diverges**.

If the limit (8.2) is infinite, we say that $\int_{-\infty}^{b} f(x)dx$ **diverges to** $-\infty/+\infty$ *and write*

$$\int_{-\infty}^{b} f(x)dx = -\infty, \quad \text{or} \quad \int_{-\infty}^{b} f(x)dx = +\infty,$$

respectively.

A similar approach is used for integrals over the entire line. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as the limit of $\int_{\mu}^{\lambda} f(x)dx$, when $\mu \to -\infty$ and $\lambda \to +\infty$, independently of each other, provided that this limit exists. Equivalently, $\int_{-\infty}^{\infty} f(x)dx$ can be expressed as the sum $\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx$, where b is any real number. Since b can be arbitrarily chosen, in practice we choose b so that the two integrals $\int_{-\infty}^{b} f(x)dx$ and $\int_{b}^{+\infty} f(x)dx$ can be easily examined.

Definition 8.3 *Let $f(x)$ be bounded over the whole line, $-\infty < x < \infty$. We say that $\int_{-\infty}^{+\infty} f(x)dx$* **converges** *if*

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\mu,\lambda\to+\infty}\int_{-\mu}^{\lambda} f(x)dx.$$

Otherwise, we say that $\int_{-\infty}^{+\infty} f(x)dx$ **diverges**.

Definition 8.4 *The integral of a function $f(x)$ defined and bounded on an interval that is not bounded, such as*

$$\int_{-\infty}^{+\infty} f(x)dx, \quad \int_{-\infty}^{b} f(x)dx, \quad \text{or} \quad \int_{b}^{+\infty} f(x)dx,$$

is called an **improper integral of the first kind**.

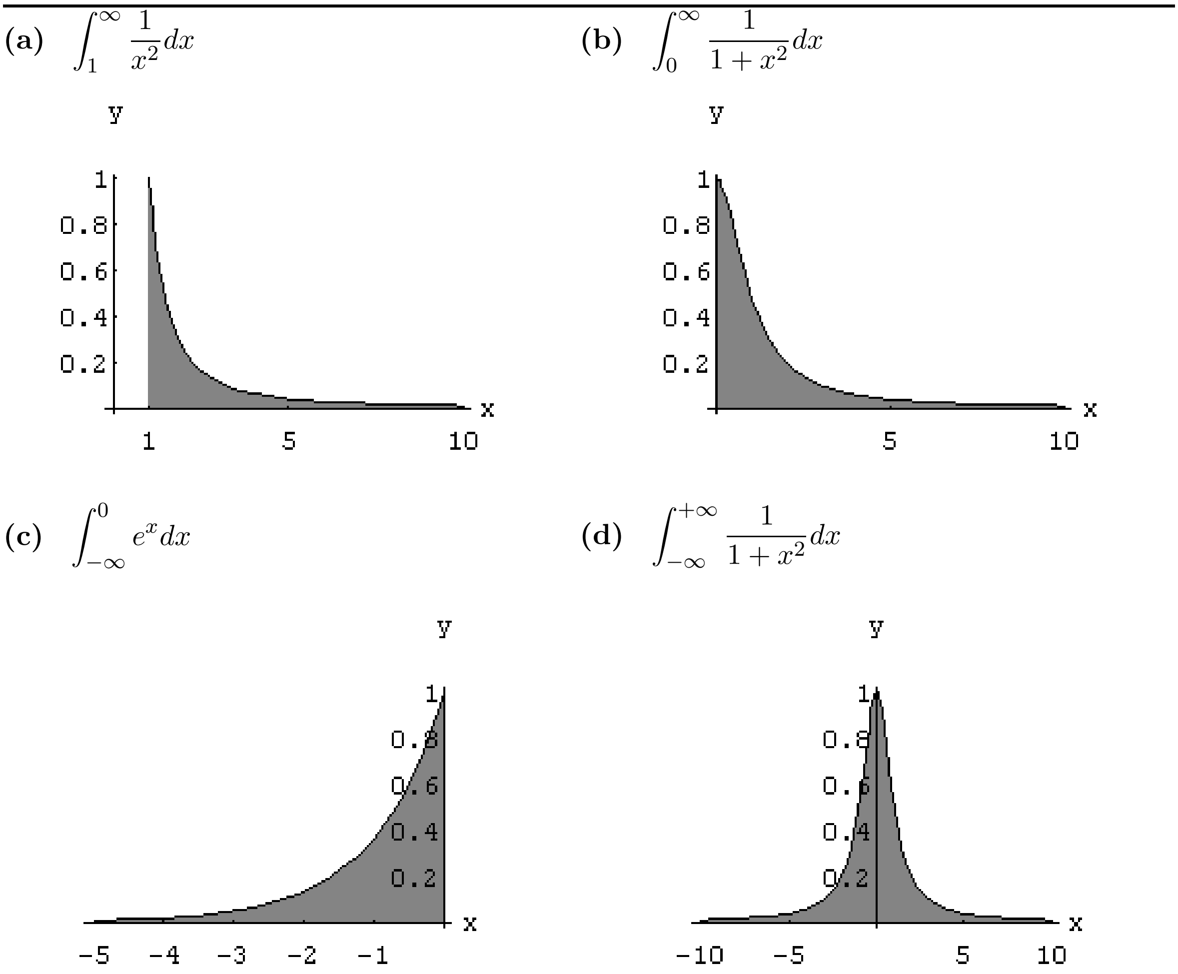


Figure 8.2: Illustrating convergence of improper integrals of the first kind.

Example 8.2 *Examining convergence of improper integrals of the first kind.*

(a) Consider the infinite integral $\int_1^\infty \frac{1}{x^2}dx$. We have

$$I(\lambda) = \int_1^\lambda \frac{1}{x^2}dx = \left(-\frac{1}{x}\right)\Bigg|_1^\lambda = 1 - \frac{1}{\lambda},$$

and $\lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty}\left(1-\frac{1}{\lambda}\right) = 1$. Therefore, the infinite integral $\int_1^\infty \frac{1}{x^2}$ converges to the limit 1:

$$\int_1^\infty \frac{1}{x^2}dx = 1.$$

(b) Consider $\int_0^\infty \frac{1}{1+x^2}dx$. We have

$$I(\lambda) = \textstyle\int_0^\lambda \frac{1}{1+x^2}dx = \arctan x\Big|_0^\lambda = \arctan\lambda,$$

$$\lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty}\arctan\lambda = \frac{\pi}{2}.$$

Therefore, the infinite integral $\int_0^\infty \frac{1}{1+x^2}dx$ converges to $\frac{\pi}{2}$:

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

(c) $\int_{-\infty}^0 e^x dx = 1$, since

$$I(\lambda) = \int_{-\lambda}^0 e^x dx = e^x\Big|_{-\lambda}^0 = 1-e^{-\lambda} = 1-\frac{1}{e^\lambda} \to 1, \quad \text{as} \quad \lambda\to\infty.$$

(d) Consider $\int_1^\infty \frac{1}{\sqrt{x}}dx$.

We have

$$I(\lambda) = \int_1^\lambda \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^\lambda = 2(\sqrt{\lambda}-1) \to +\infty, \quad \text{as} \quad \lambda\to+\infty.$$

Thus $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges to $+\infty$ and we write $\int_1^\infty \frac{1}{\sqrt{x}}dx = +\infty$.

(e) Consider $\int_{-\infty}^0 \sin x dx$. Since

$$I(\lambda) = \int_{-\lambda}^0 \sin x dx = -\cos x\Big|_{-\lambda}^0 = -1+\cos\lambda,$$

$I(\lambda)$ has no limit as $\lambda\to\infty$ ($I(\lambda)$ oscillates between -1 and 1), we conclude that $\int_{-\infty}^0 \sin x dx$ diverges.

(f) The integral $\int_0^\infty \sin x dx$ diverges, since

$$I(\lambda) = \int_0^\lambda \sin x dx = -\cos x\Big|_0^\lambda = 1-\cos\lambda,$$

has no limit as $\lambda\to\infty$.

(g) The integral $\int_{-\infty}^{0} \frac{1}{1+x^2}dx$ converges to $\frac{\pi}{2}$, since

$$I(\lambda) = \int_{-\lambda}^{0} \frac{1}{1+x^2}dx = \arctan x\Big|_{-\lambda}^{0} = -\arctan(-\lambda) = \arctan\lambda \to \frac{\pi}{2}, \quad \text{as} \quad \lambda \to +\infty.$$

(h) Consider $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$. We have

$$\begin{aligned} \int_{-\mu}^{\lambda} \frac{1}{1+x^2}dx &= \arctan\lambda + \arctan\mu, \\ \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \lim_{\lambda,\mu\to+\infty}(\arctan\lambda + \arctan\lambda) = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \end{aligned}$$

(i) The integral $\int_{-\infty}^{\infty} \sin x dx$ diverges, since

$$\int_{-\infty}^{\infty} \sin x dx = \lim_{\mu,\ \lambda\to+\infty} \int_{-\mu}^{\lambda} \sin x dx = \lim_{\mu,\ \lambda\to+\infty}(\cos\mu - \cos\lambda),$$

which does not exist.

Definition 8.5 *Let f be bounded over the entire line and integrable over the interval $(-\lambda, \lambda)$, for any $\lambda > 0$. The limit*

$$\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} f(x)dx,$$

if it exists, is called the **Cauchy principal value** *of the integral* $\int_{-\infty}^{\infty} f(x)dx$, *and is written*

$$P\int_{-\infty}^{\infty} f(x)dx = \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} f(x)dx.$$

Note that the existence of the Cauchy principal value

$$\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} f(x)dx$$

does not imply that $\int_{-\infty}^{\infty} f(x)dx$ converges. The convergence of $\int_{-\infty}^{\infty} f(x)dx$ requires existence of the limit of $\int_{-\mu}^{\lambda} f(x)dx$, when $\mu \to \infty$ and $\lambda \to \infty$, independently on each other.

Example 8.3 *Find Cauchy principal value of the integral* $\int_{-\infty}^{\infty} \sin x dx$.

Solution We have

$$\int_{-\lambda}^{\lambda} \sin x dx = -\cos x\Big|_{-\lambda}^{\lambda} = -\cos\lambda + \cos\lambda = 0,$$

so that

$$\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\sin x dx = 0$$

and

$$P\int_{-\infty}^{\infty}\sin x dx = 0. \blacksquare$$

Note, that we have already observed (see the previous example **(i)**) that $\int_{-\infty}^{\infty}\sin x dx$ does not exist.

8.3 Improper Integrals of the Second Kind

In this section we consider integrals over a finite interval $[a, b]$, when the integrand $f(x)$ has infinite singularity at some point or points in $[a, b]$. Recall that f has an infinite singularity at the point $x = x_o$, if f is not defined at the point x_o and when one-sided limits, as x approaches x_o, are infinite.

Definition 8.6 *Assume that f is integrable on every interval of the form $[a, b-\varepsilon]$, where $0 < \varepsilon < b - a$, but f has an infinite singularity at the point $x = b$, that is $\lim_{x\to b-} f(x)$ is infinite.*

Then the integral $\int_a^b f(x)$ is called an **improper integral** *and is defined as*

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+}\int_a^{b-\varepsilon} f(x)dx,$$

provided that this limit exists.

Definition 8.7 *Assume that f is integrable on every interval of the form $[a+\varepsilon, b]$, where $0 < \varepsilon < b - a$, but f has an infinite singularity at the point $x = a$, that is $\lim_{x\to a+} f(x)$ is infinite.*

Then the integral $\int_a^b f(x)$ is called an **improper integral** *and is defined as*

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+}\int_{a+\varepsilon}^{b} f(x)dx,$$

provided that this limit exists.

If the appropriate limit in Definitions 8.6 and 8.7 exists, we say that the improper integral $\int_a^b f(x)dx$ **converges**. Otherwise, we say that $\int_a^b f(x)dx$ **diverges**.

Definition 8.8 *Suppose that f has an infinite singularity at a point $x = x_o$, where $a < x_o < b$. We say that the improper integral $\int_a^b f(x)dx$* **converges** *if and only if both improper integrals $\int_a^{x_o} f(x)dx$ and $\int_{x_o}^b f(x)dx$ converge. Otherwise we say that the improper integral $\int_a^b f(x)dx$* **diverges**.

If $\int_a^b f(x)dx$ *converges, then, we define:*

$$\int_a^b f(x)dx = \int_a^{x_o} f(x)dx + \int_{x_o}^b f(x)dx.$$

The integrals considered in Definitions 8.6, 8.7, and 8.8 are called improper integrals of the second kind.

Definition 8.9 *The integral* $\int_a^b f(x)dx$ *over the finite interval* $[a, b]$ *is said to be an* **improper integral of the second kind** *if the integrand* $f(x)$ *has an infinite singularity at finitely many points of the interval* $[a, b]$.

The case when $f(x)$ has more than one singularity point in $[a, b]$ is illustrated in the next example **(f)**.

Example 8.4 *Examining convergence of improper integrals of the second kind.*

(a) Consider $\int_0^1 \frac{1}{\sqrt{x}}dx$. The integrand $f(x) = \frac{1}{\sqrt{x}}$ is not bounded on the interval $(0, 1]$ and $\lim_{x\to 0-} f(x) = +\infty$. The function $f(x)$, however, is integrable over every interval $[\varepsilon, 1]$, for $0 < \varepsilon < 1$. We have

$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_\varepsilon^1 = 2(1-\sqrt{\varepsilon}) \to 2, \quad \text{as} \quad \varepsilon \to 0+.$$

Therefore, the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges to the limit 2:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2.$$

(b) Consider $\int_0^1 \frac{1}{x}dx$. Similarly to **(a)**, the integrand $f(x) = \frac{1}{x}$ is not bounded on the interval $(0, 1]$ and $\lim_{x\to 0-} f(x) = +\infty$. Moreover, the function $f(x)$ is integrable over every interval $[\varepsilon, 1]$, for $0 < \varepsilon < 1$. Now we have

$$\int_\varepsilon^1 \frac{1}{x}dx = \ln x\Big|_\varepsilon^1 = -\ln\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0+.$$

Therefore, the improper integral $\int_0^1 \frac{1}{x}dx$ diverges to $+\infty$ and we write

$$\int_0^1 \frac{1}{x}dx = +\infty.$$

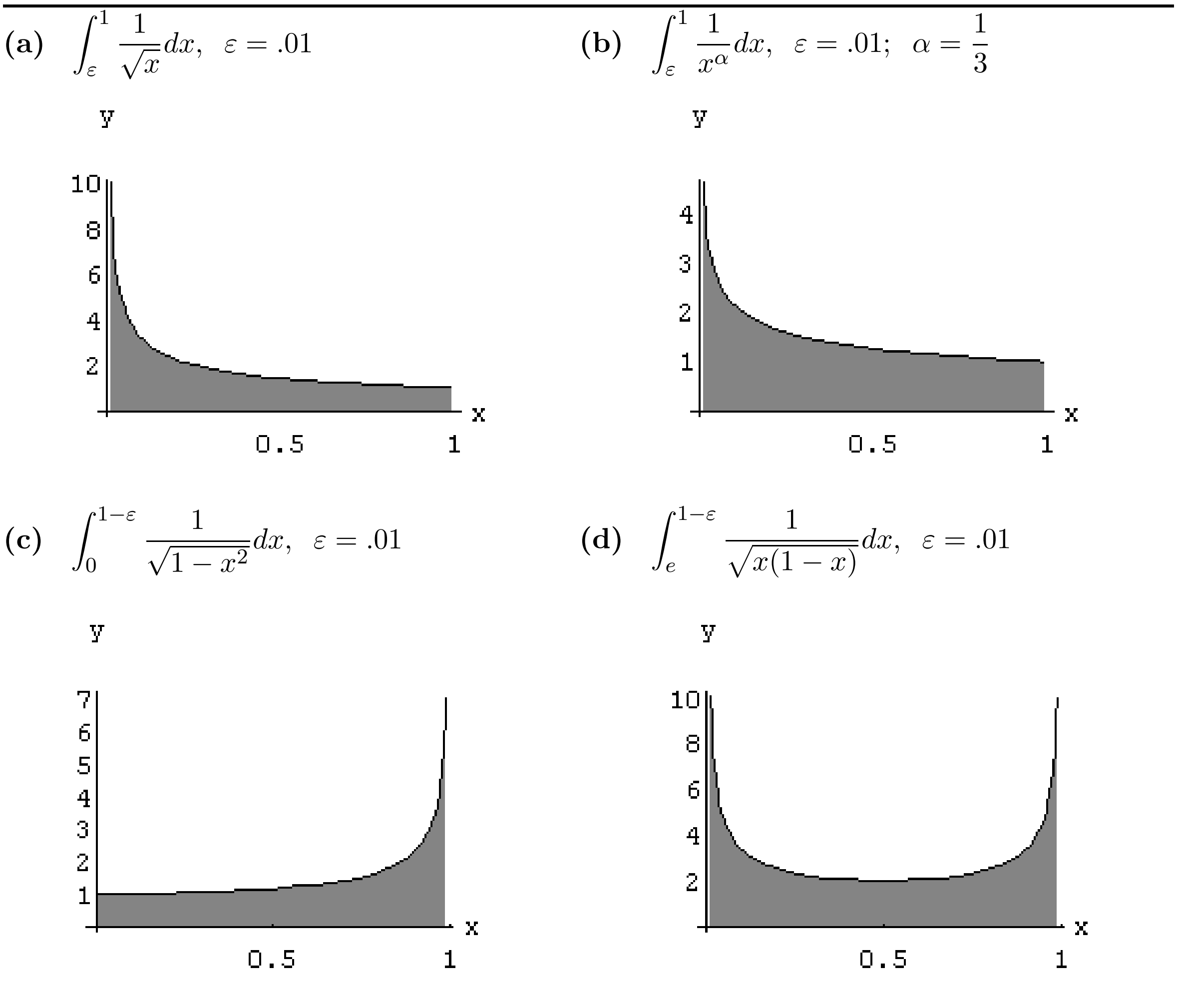


Figure 8.3: Illustrating convergence of improper integrals of the second kind.

(c) Consider $\displaystyle\int_0^1 \frac{1}{x^\alpha} dx$, where α is any real number. We note that we have already considered this integral in **(a)** with $\alpha = \frac{1}{2}$ and in **(b)** with $\alpha = 1$.

We deal with the infinite singularity at the point $a = 0$, and we note that the function $f(x) = \dfrac{1}{x^\alpha}$ is integrable over every interval $[\varepsilon, 1]$ for $0 < \varepsilon < 1$ and for every real value of α.

We have

$$\int_{\varepsilon}^{1} \frac{1}{x^\alpha} dx = \frac{1}{1-\alpha} x^{(1-\alpha)} \Big|_{\varepsilon}^{1} = \frac{1}{1-\alpha} \left(1 - \varepsilon^{1-\alpha}\right),$$

provided $\alpha \neq 1$. Thus

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{1}{x^\alpha} dx = \begin{cases} \dfrac{1}{1-\alpha} & \text{if } \alpha < 1 \\ +\infty & \text{if } \alpha > 1. \end{cases}$$

Combining the above with the result obtained in **(b)**, we conclude that the integral $\int_0^1 \frac{1}{x^\alpha}dx$ converges for $\alpha < 1$ and diverges for $\alpha \geq 1$. If $\alpha < 1$ then

$$\int_0^1 \frac{1}{x^\alpha}dx = \frac{1}{1-\alpha}.$$

(d) Let $f(x) = \frac{1}{\sqrt{1-x^2}}$ and consider the integral $\int_0^1 f(x)dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx$. We note that $\lim_{x\to 1-} f(x) = +\infty$, so that $f(x)$, defined for $|x| < 1$, is not bounded on the interval $[0,1)$ (see Figure 8.3(c)). The integral has a singularity point at $x = 1$, but $f(x)$ is integrable over any interval $[0, 1-\varepsilon]$, where $0 < \varepsilon < 1$. We have

$$\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}dx = \arcsin x\Big|_0^{1-\varepsilon} = \arcsin(1-\varepsilon) \to \frac{\pi}{2}, \quad \text{as} \quad \varepsilon \to 0+.$$

Hence $\int_0^1 f(x)dx$ converges to $\pi/2$:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}.$$

(e) Consider the integral $\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx$. It has two singularity points: at $x = 0$ and at $x = 1$ (refer to Figure 8.3(d)).

We have

$$\int_{\varepsilon'}^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}dx = \arcsin(2x-1)\Big|_{\varepsilon'}^{1-\varepsilon} = \arcsin(1-2\varepsilon) - \arcsin(2\varepsilon'-1).$$

Thus

$$\begin{aligned}
\lim_{\varepsilon,\varepsilon'\to 0+} \int_{\varepsilon'}^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}dx &= \lim_{\varepsilon\to 0+} \arcsin(1-2\varepsilon) - \lim_{\varepsilon'\to 0+} \arcsin(2\varepsilon'-1) \\
&= \arcsin 1 - \arcsin(-1) \\
&= \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi.
\end{aligned}$$

Therefore we conclude that the improper integral $\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx$ converges to π:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx = \lim_{\varepsilon,\varepsilon'\to 0+} \int_{\varepsilon'}^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}dx = \pi.$$

(f) Consider the integral $\int_0^1 \frac{1}{1-x}dx$ that has a singularity point at $x = 1$: $\lim_{x\to 1-} \frac{1}{1-x} = +\infty$. We have

$$\int_0^{1-\varepsilon} \frac{1}{1-x}dx = -\ln(1-x)\Big|_0^{1-\varepsilon} = -\ln\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0+.$$

Therefore, $\int_0^1 \frac{1}{1-x}dx$ diverges to $+\infty$:

$$\int_0^1 \frac{1}{1-x}dx = +\infty.$$

(a) $\lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{x}dx = +\infty$ **(b)** $\lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{1}{1-x}dx = +\infty$

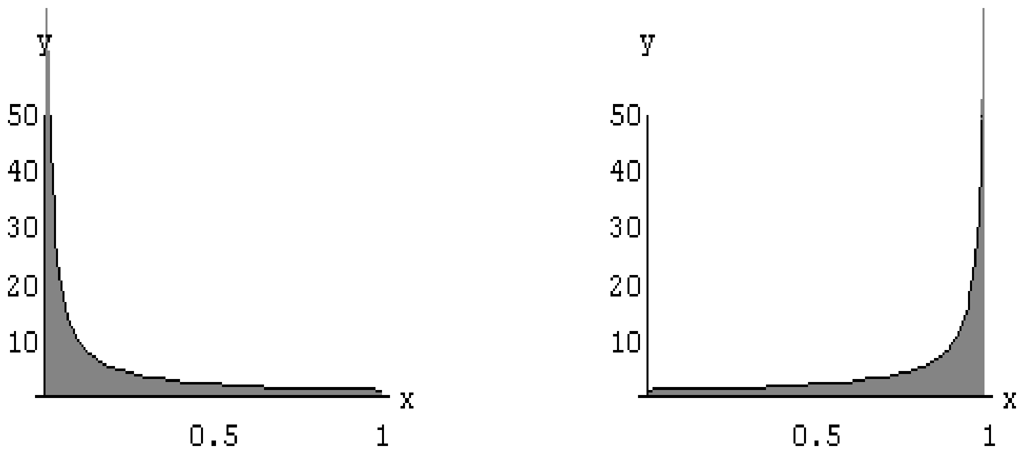


Figure 8.4: Illustrating divergence of improper integrals to $+\infty$.

8.4 Comparison Tests for Integrals

Theorem 8.1 Comparison Test I

Suppose that f and g are defined on the interval $[a, +\infty)$ and integrable on $[a, \lambda]$ for every $\lambda > a$.

If

$$0 \leq f(x) \leq g(x),$$

for all $x \in [a, +\infty)$, then

(i) $\int_a^{+\infty} f(x)dx$ *converges if* $\int_a^{+\infty} g(x)dx$ *converges;*

(ii) $\int_a^{+\infty} g(x)dx$ *diverges if* $\int_a^{+\infty} f(x)dx$ *diverges.*

Proof. Since $0 < f(x) < g(x)$, for $x \geq a$, we get

$$0 \leq \int_a^{\lambda} f(x)dx \leq \int_a^{\lambda} g(x)dx, \quad x \geq a,$$

and each integral is monotone increasing function of λ.

Hence, if $\int_a^{\lambda} g(x)dx$ converges, $\int_a^{\lambda} f(x)dx$ is bounded above and so it converges.

If $\int_a^{\lambda} f(x)dx$ diverges then $\int_a^{\lambda} g(x)dx$ is unbounded and hence diverges. ∎

Example 8.5 *Does* $\int_0^{+\infty} \frac{1}{e^x + 3}dx$ *converge?*

Solution. Let $f(x) = e^{-x}$ and $g(x) = \dfrac{1}{e^x + 3}$, for $x \in [0, +\infty)$. We have

$$0 < \frac{1}{e^x + 3} < \frac{1}{e^x} = e^{-x}, \quad x \in [0, +\infty),$$

and both functions, f and g, are integrable on $[0, \lambda]$ for every $\lambda > 0$. Thus the hypotheses of the Comparison Test I are satisfied. Now,

$$\int_0^{+\infty} f(x)dx = \int_0^{+\infty} e^{-x}dx = -\lim_{\lambda \to +\infty} e^{-x} = -\lim_{\lambda \to +\infty} e^{-x}\Big|_0^{\lambda} = -\lim_{\lambda \to +\infty} (1 - e^{-\lambda}) = 1.$$

Therefore, by the Comparison Test, the improper integral $\int_0^{+\infty} \frac{1}{e^x + 3}dx$ converges. ∎

An analogous comparison test holds for improper integrals of the second kind. We leave its formulation to the reader. The following example illustrates the point.

Example 8.6 *Does* $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ *converge?*

Solution. We have

$$\frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}, \quad x \in (0,1],$$

and

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to 0+} 2\sqrt{x}\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0+}(2-2\sqrt{\varepsilon}) = 2.$$

Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, by the Comparison Test, we conclude that $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ converges.

8.5 Improper Integrals of the Third Kind

In this section we consider integrals of an unbounded function $f(x)$ over an infinite interval, such as

$$\int_{-\infty}^b f(x)dx, \quad \int_a^{+\infty} f(x)dx, \quad \text{or} \quad \int_{-\infty}^{+\infty} f(x)dx,$$

where the integrand $f(x)$ has one (or more) singularity points in the interval of integration. This kind of integral is called an **improper integral of the third kind**.

In order to examine convergence of an integral of this type, we express it as the sum of two improper integrals; one of the first kind and the other of the second kind.

By definition, we say that the improper integral of the third kind converges if and only if the corresponding integrals of the first and the second kind both converge.

Example 8.7 *Examining convergence of an improper integral of the third kind.*

Consider the infinite integral $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ and note that the integrand has a singular point at $x=0$. Thus $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ is an improper integral of the third kind.

We can write

$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx = \int_0^b \frac{1}{x^2+\sqrt{x}}dx + \int_b^\infty \frac{1}{x^2+\sqrt{x}}dx,$$

where the point $x=b$ for splitting up the interval of integration can be chosen quite arbitrarily.

Let $b=1$. We examine separately convergence of each of

$$\int_0^1 \frac{1}{x^2+\sqrt{x}}dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^2+\sqrt{x}}dx.$$

The integral $\int_0^1 \frac{1}{x^2+\sqrt{x}}dx$ converges by Example 8.6.

The integral $\int_1^\infty \frac{1}{x^2+\sqrt{x}}dx$ converges, by the Comparison Test, because $\frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}$, $x \in [1,\infty)$ and $\int_1^\infty \frac{1}{x^2}dx$ converges.

Therefore, we conclude that the integral

$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$$

converges. ∎

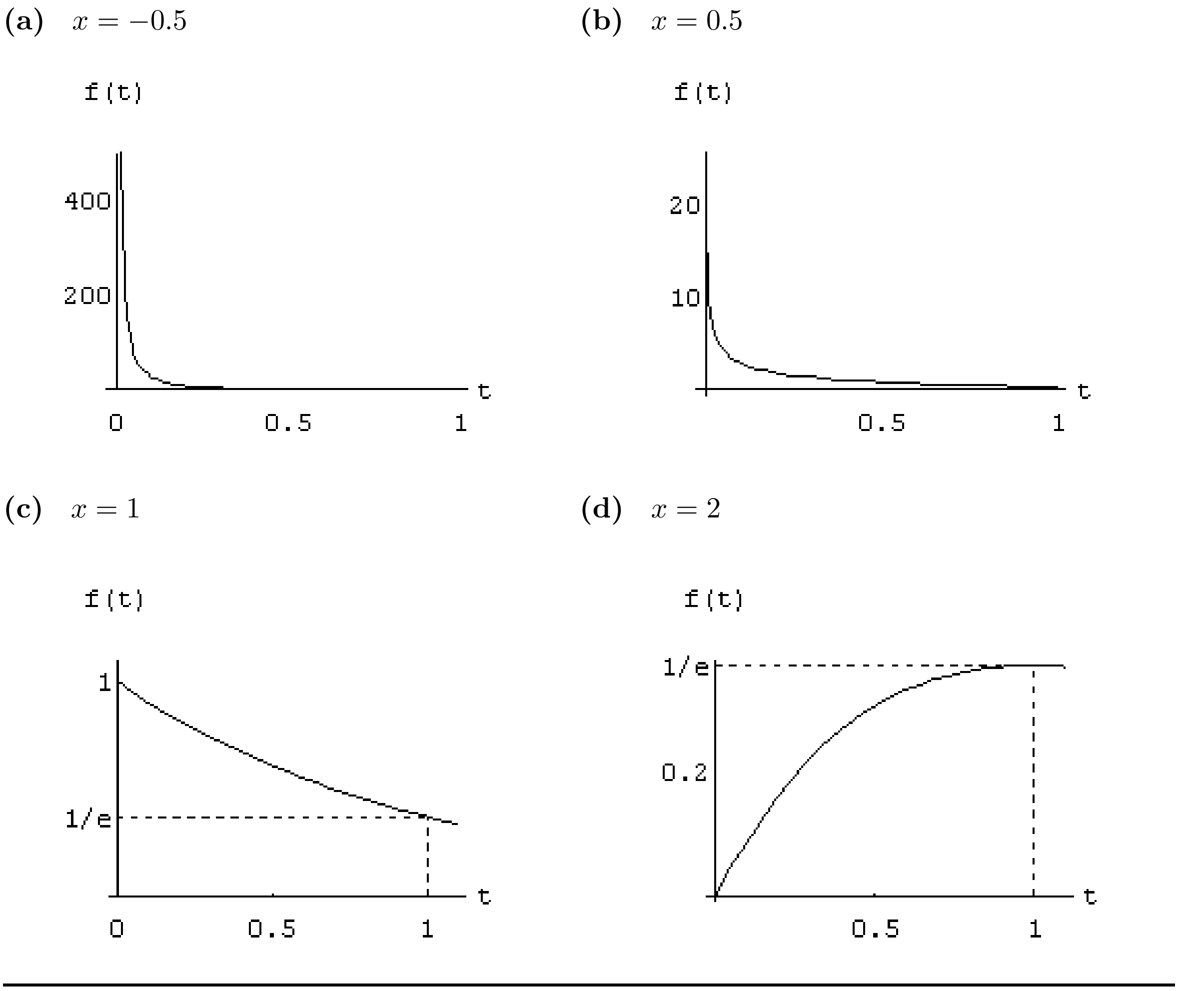


Figure 8.5: The function $f(t) = t^{x-1}e^{-t}$, $0 < t < 1$, for selected values of x.

8.6 The Gamma Function

Example 8.8 *Is the function $f(t) = t^{x-1}e^{-t}$, where x is a real number, integrable over the interval $[0,1]$?*

Solution. We shall consider separately the three cases: $x \leq 0$, $0 < x < 1$, and $x \geq 1$. Refer to Figure 8.5 to see graphs of $f(t)$, $0 < t < 1$, for selected values of x.

Case 1: $x \leq 0$.

On the interval $[0,1]$ we have $e^t < 3$, so $t^{x-1}e^{-t} > \dfrac{1}{3}t^{x-1}$. By Example 8.4(b), $\displaystyle\int_0^1 t^{x-1}dt$ diverges, since $x - 1 \leq -1$. Hence $\displaystyle\int_0^1 t^{x-1}e^{-t}dt$ diverges for $x \leq 0$.

Case 2: $0 < x < 1$.

If $0 < x < 1$, then the function $f(t) = t^{x-1}e^{-t}$ has an infinite singularity at the point $t = 0$ and $\displaystyle\int_0^1 t^{x-1}e^{-t}dt$ is an improper integral of the second kind.

If $t \geq 0$ then $0 < e^{-t} \leq 1$ and $t^{x-1}e^{-t} \leq t^{x-1}$. Now, the improper integral $\displaystyle\int_0^1 t^{x-1}dt$ converges for all values of x such that $0 < 1-x < 1$ or $0 < x < 1$. By the Comparison Test, therefore, we conclude that the improper integral $\int_0^1 t^{x-1}e^{-t}dt$ converges for $0 < x < 1$.

Case 3: $x \geq 1$.

The function $f(t)$ is continuous for $t \in [0,1]$ and, therefore, integrable. Hence $t^{x-1}e^{-t}dt$ exists for $x \geq 1$.

Therefore, the function $f(t) = t^{x-1}e^{-t}$ is integrable over the interval $[0,1]$, provided that $x > 0$, but not integrable if $x \leq 0$. $\blacksquare$

Example 8.9 *Is the function $f(t) = t^{x-1}e^{-t}$, where $x > 0$, integrable on the interval $[1, +\infty)$?*

Solution. The integral $\displaystyle\int_1^{+\infty} t^{x-1}e^{-t}dt$ is an improper integral of the first kind. We shall prove that it converges by comparing the integrand $f(x)$ with the function $g(t) = t^{-2}$.

We have

$$\lim_{t\to+\infty} \frac{t^{x+1}}{e^t} = 0,$$

which implies that

$$\forall \varepsilon > 0 \ \exists M \quad \left(t \geq M \quad \Longrightarrow \quad \frac{t^{x+1}}{e^t} \leq \varepsilon\right).$$

Let $\varepsilon = 1$ and denote by t_o the corresponding value of M. Then we have

$$\frac{t^{x+1}}{e^t} \leq 1 \quad \text{or} \quad e^{-t}t^{x-1} \leq t^{-2} \quad \text{for} \quad t > t_o.$$

Now, the improper integral $\int_{t_o}^{+\infty} t^{-2}dt$ converges. Hence, by the Comparison Test, $\int_{t_o}^{+\infty} e^{-t}t^{x-1}dt$ converges.

Since $f(t) = e^{-t}t^{x-1}$ is integrable on any interval of the form $[1, t_o]$, we conclude that

$$\int_1^{+\infty} t^{x-1}e^{-t}dt = \int_1^{t_o} t^{x-1}e^{-t}dt + \int_{t_o}^{+\infty} t^{x-1}e^{-t}dt$$

converges, when $x > 0$.

Therefore, the function $f(t) = t^{x-1}e^{-t}$, where $x > 0$, is integrable on the interval $[1, +\infty)$.

∎

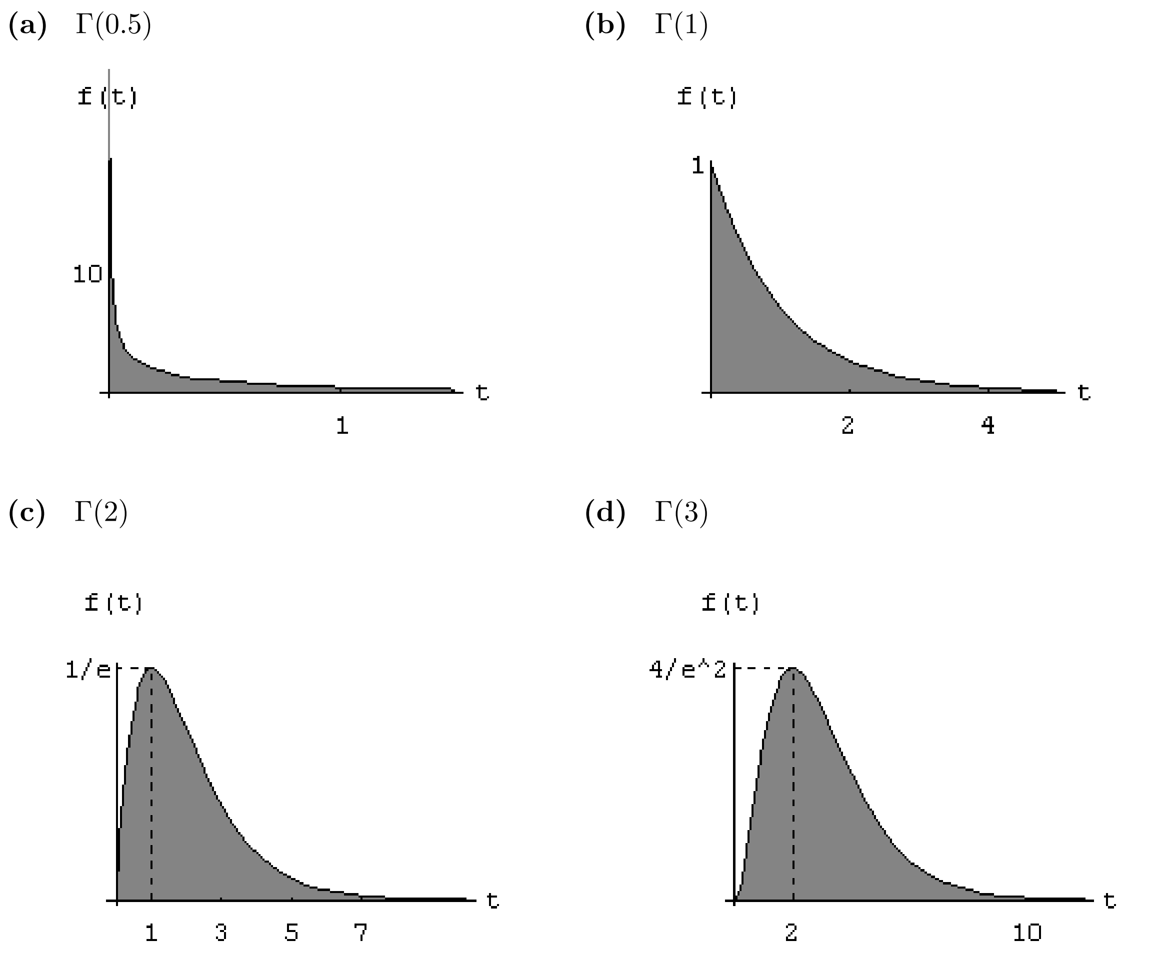


Figure 8.6: The Gamma function $\Gamma(x)$ as the area under the curve $y = t^{x-1}e^{-t}$ for fixed values of x.

Example 8.10 *Show that the integral*

$$\int_0^{\infty} e^{-t}t^{x-1}dt, \tag{8.3}$$

converges for all positive values of x.

Solution.
Case 1. $0 < x < 1$.
The integrand $f(t) = e^{-t}t^{x-1}$ has an infinite singularity at the point $t = 0$ and (8.3) is an improper integral of the third kind. Refer to Figure 8.6(a).

We choose the point b for splitting up the interval of integration to be $b = 1$. Then we write

$$\int_0^{+\infty} e^{-t}t^{x-1}dt = I_1 + I_2 = \int_0^1 e^{-t}t^{x-1}dt + \int_1^{+\infty} e^{-t}t^{x-1}dt.$$

By Example 8.8 and Example 8.9, the improper integrals I_1 and I_2 converge. Hence we conclude that the integral (8.3) converges, when $0 < x < 1$.

Case 2. $x \geq 1$.
The integrand $f(t) = t^{x-1}e^{-t}$ is bounded at the point $t = 0$ and (8.3) is an improper integral of the first kind. Refer to Figure 8.6 (b), (c) and (d), which gives us the approximate values of $\int_0^{\infty} e^{-t}t^{x-1}dt$ (as the shaded area) for $x = 1$, $x = 2$, and $x = 3$. As we already discussed in Example 8.8, $f(t)$ is integrable over the interval $[0, 1]$. Now, by example 8.9, the improper integral $\displaystyle\int_1^{+\infty} t^{x-1}e^{-t}dt$ converges, when $x \geq 1$. Hence

$$\int_0^{+\infty} e^{-t}t^{x-1}dt = \int_0^1 e^{-t}t^{x-1}dt + \int_1^{+\infty} e^{-t}t^{x-1}dt$$

converges when $x \geq 1$. ∎

The last example justifies the existence of the improper integral

$$\int_0^{+\infty} t^{x-1}e^{-t}dt$$

for any positive value of x. This integral considered as a function of the variable x, $x > 0$, is called the gamma function and denoted by the symbol $\Gamma(x)$.

Note that the integral diverges for $x \leq 0$.

Definition 8.10 *The* **gamma function** *is defined on the interval* $(0, +\infty)$ *as*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad x > 0. \tag{8.4}$$

In Figure 8.7(a), five instances of the integrand $f(t) = t^{x-1}e^{-t}$ are plotted over the interval $t \in (0, 10)$, namely $f(t)$ corresponding to the following values of x: $0.5, 1, 2, 3$. Figure 8.7(b) shows us the gamma function $\Gamma(x)$ for $0 < x \leq 5$.

(a) $f(t) = t^{x-1}e^{-t}$ **(b)** $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$

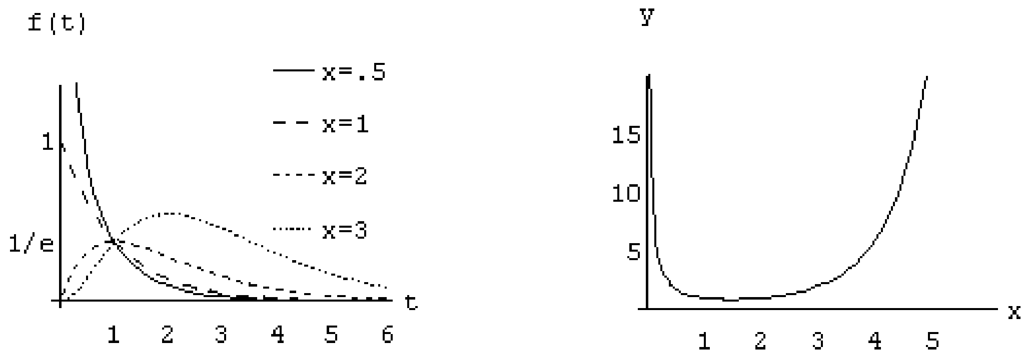


Figure 8.7: **(a)** The function $f(t) = t^{x-1}e^{-t}$, $0 < t < 6$, for selected values of x; **(b)** The Gamma function $\int_0^{\infty} t^{x-1}e^{-t}dt$, $0 < x < 5$.

Theorem 8.2 *The function $\Gamma(x)$ defined by (8.4) has the property*

$$\Gamma(x+1) = x\Gamma(x). \tag{8.5}$$

Proof. We have

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}dt = \lim_{\lambda\to+\infty}\int_0^{\lambda} t^x e^{-t}dt.$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^{\lambda} t^x e^{-t}dt &= \left.\frac{-t^x}{e^t}\right|_{t=0}^{t=\lambda} + \int_0^{\lambda} x t^{x-1}e^{-t}dt \\ &= -\frac{\lambda^x}{x^{\lambda}} + x\int_0^{\lambda} t^{x-1}e^{-t}dt.\end{aligned}$$

Hence

$$\begin{aligned}\Gamma(x+1) &= \lim_{\lambda\to+\infty}\left(-\frac{\lambda^x}{e^{\lambda}}\right) + x\lim_{\lambda\to+\infty}\int_0^{\lambda} t^{x-1}e^{-t}dt \\ &= 0 + x\int_0^{+\infty} t^{x-1}e^{-t}dt \\ &= x\Gamma(x). \quad \blacksquare\end{aligned}$$

Example 8.11 *Show that* $\Gamma(n) = (n-1)!$, *if* n *is a positive integer.*

Solution.
We apply the principle of mathematical induction.
Let $n = 1$. Then

$$\Gamma(1) = \int_0^{+\infty} e^{-t}dt = \lim_{\lambda\to+\infty} -e^{-t}\Big|_0^{\lambda} = 1 = 0!.$$

Assume that $\Gamma(n) = (n-1)!$. Then , by 8.5,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!. \quad \blacksquare$$

8.7 Absolute Convergence of Improper Integrals

Definition 8.11 *If* f *is bounded and integrable over any interval* $[a,\lambda]$ *with* $\lambda > 0$, *and* $\displaystyle\int_a^{+\infty} |f(x)|dx$ *converges, then we say that the integral* $\displaystyle\int_a^{+\infty} f(x)dx$ *is* **absolutely convergent**.

Let

$$I(\lambda) = \int_a^{\lambda} f(x)dx.$$

If $\lambda_1 > \lambda_1$ then

$$I(\lambda_2) - I(\lambda_1) = \int_{\lambda_1}^{\lambda_2} f(x)dx.$$

Similarly to the Cauchy criterion for convergence of sequences, we have the following **Cauchy Condition** for convergence of improper integrals.

A necessary and sufficient condition for the convergence of $I(\lambda)$ as $\lambda \to +\infty$ is the Cauchy condition:

$$\forall\varepsilon > 0 \;\; \exists M \quad \left(\lambda_2 > \lambda_1 > M \quad \Longrightarrow \quad \left|\int_{\lambda_1}^{\lambda_2} f(x)dx\right| < \varepsilon\right). \tag{8.6}$$

Theorem 8.3 *Assume that* $f(x)$ *is integrable on every interval* (a,λ), $\lambda > a$. *If* $\displaystyle\int_a^{\infty} |f(x)|\,dx$ *converges, then* $\displaystyle\int_a^{\infty} f(x)\,dx$ *converges.*

Proof. We assume that $\int_a^{+\infty} |f(x)|dx$ converges so, using the Cauchy general condition for convergence, we have

$$\forall\varepsilon > 0 \;\; \exists M > a \quad \left(\lambda_2 > \lambda_1 > M \quad \Longrightarrow \quad \left|\int_{\lambda_1}^{\lambda_2} |f(x)|dx\right| < \varepsilon\right).$$

Now

$$\left|\int_{\lambda_1}^{\lambda_2} f(x)dx\right| \le \int_{\lambda_1}^{\lambda_2} |f(x)|dx,$$

and we conclude that

$$\forall \varepsilon > 0 \ \exists M > a \quad \left(\lambda_2 > \lambda_1 > M \quad \Longrightarrow \quad \left| \int_{\lambda_1}^{\lambda_2} f(x)dx \right| < \varepsilon \right),$$

which means that $\int_a^{+\infty} f(x)dx$ converges. $\blacksquare$

Example 8.12 *Show that* $\displaystyle\int_0^{+\infty} \frac{\cos x}{1+x^2} dx$ *converges absolutely.*

Solution. We have

$$\left| \frac{\cos x}{1+x^2} \right| \leq \frac{1}{1+x^2}.$$

Since $\displaystyle\int_0^{\infty} \frac{1}{1+x^2} dx$ converges, by the Comparison test we conclude that $\displaystyle\int_0^{+\infty} \left| \frac{\cos x}{1+x^2} \right| dx$ converges. $\blacksquare$

Example 8.13 *Show that the improper integral* $\displaystyle\int_0^{+\infty} \frac{\sin x}{x} dx$ *is convergent but not absolutely convergent.*

Solution.

(i) To show that the integral converges we shall show that the Cauchy general condition for convergence is satisfied:

$$\forall \varepsilon > 0 \ \exists M \quad \left(\lambda_2 > \lambda_1 > M \quad \Longrightarrow \quad \left| \int_{\lambda_1}^{\lambda_2} \frac{\sin x}{x} dx \right| < \varepsilon \right). \tag{8.7}$$

By the (second) Mean Value Theorem for integrals with

$$f(x) = \frac{1}{x}, \quad g(x) = \sin x,$$

we have

$$\int_{\lambda_1}^{\lambda_2} \frac{\sin x}{x} dx = \frac{1}{\lambda_1} \int_{\lambda_1}^{\xi} \sin x dx + \frac{1}{\lambda_2} \int_{\xi}^{\lambda_2} \sin x dx,$$

for some ξ, $0 < \lambda_1 < \xi < \lambda_2$. Now, for any $[a, b]$, we have

$$\left| \int_a^b \sin x dx \right| = |\cos a - \cos b| \leq |\cos a| + |\cos b| \leq 2.$$

Hence

$$\left| \int_{\lambda_1}^{\lambda_2} \frac{\sin x}{x} dx \right| \leq \frac{1}{\lambda_1} \left| \int_{\lambda_1}^{\xi} \sin x dx \right| + \frac{1}{\lambda_2} \left| \int_{\xi}^{\lambda_2} \sin x dx \right| \leq 2 \left| \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right| \leq \frac{4}{\lambda_1} < \frac{4}{M},$$

provided $\lambda_2 > \lambda_1 > M$. Therefore we have shown that, given any $\varepsilon > 0$, there is $M = \dfrac{4}{\varepsilon}$ such that, if $\lambda_2 > \lambda_1 > M$ then

$$\left| \int_{\lambda_1}^{\lambda_2} \frac{\sin x}{x} dx \right| < \varepsilon.$$

(ii) To examine absolute convergence of the integral $\int_0^{+\infty} \frac{\sin x}{x} dx$, we note that

$$\int_0^{\lambda} \frac{|\sin x|}{x} dx > \int_0^{n\pi} \frac{|\sin x|}{x} dx,$$

where n is any natural number such that $\lambda > n\pi$.

Now,

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} dy,$$

by substitution of $x = y + (r-1)\pi$.

Since

$$\frac{1}{(r-1)\pi + y} \geq \frac{1}{r\pi}, \quad 0 \leq y \leq \pi,$$

we obtain

$$\int_0^{\lambda} \frac{|\sin x|}{x} dx > \sum_{r=1}^{n} \frac{1}{r\pi} \int_0^{\pi} \sin y dy = \frac{2}{\pi} \sum_{r=1}^{n} \frac{1}{r}.$$

We know that the series $\sum_{r=1}^{n} \frac{1}{r}$ is unbounded, which implies that $\int_0^{+\infty} \frac{|\sin x|}{x} dx$ is unbounded and, consequently, the integral $\int_0^{+\infty} \frac{\sin x}{x}$ is not absolutely convergent. ∎

8.8 Derivatives of Functions Defined by Integrals

Suppose f is a function of two variables (t, x), such that for each x, $a \leq x \leq b$, the function $\Phi(t, x)$ is Riemann integrable on $[\alpha, \beta]$. Then

$$F(x) = \int_{\alpha}^{\beta} f(t, x) dt$$

defines a function F on $[a, b]$.

Since integrals are limits of sums and because the derivative of a sum is the sum of derivatives, one might expect that a similar result would hold for integrals:

$$\frac{dF}{dx} = \frac{d}{dx} \left(\int_{\alpha}^{\beta} f(t, x) dt \right) = \int_{\alpha}^{\beta} \frac{\partial f}{\partial x} dt. \tag{8.8}$$

Example 8.14 *Verifying formula* (8.8).

Let

$$F(x) = \int_0^{\pi} \sin xt dt, \quad x > 0.$$

By direct integration, we obtain

$$F(x) = -\frac{\cos xt}{x} \Big|_0^{\pi} = \frac{1}{x}(1 - \cos x\pi).$$

Hence

$$F'(x) = \frac{\pi}{x} \cdot \sin \pi x - \frac{1}{x^2}(1 - \cos \pi x).$$

Note that

$$\begin{aligned}\int_0^\pi \frac{\partial}{\partial x}(\sin xt)dt &= \int_0^\pi t \cos xt \cdot dt = \frac{t}{x} \sin xt \Big|_0^\pi - \frac{1}{x} \int_0^\pi \sin xt dt \\ &= \frac{\pi}{x} \sin \pi x - \frac{1}{x^2}(-\cos xt)\Big|_0^\pi = \frac{\pi}{x} \sin \pi x - \frac{1}{x^2}(1 - \cos \pi x).\end{aligned}$$

We have indeed verified that the formula (8.8) holds. ∎

Example 8.15 *Verifying that the formula* (8.8) *does not always hold.*

It will be shown later that

$$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}, \quad \text{for all} \quad a > 0.$$

Thus $\frac{d}{da}\left(\int_0^\infty \frac{\sin ax}{x} dx\right) = 0, \quad \text{for} \quad a > 0.$

But $\int_0^\infty \frac{\partial}{\partial a}\left(\frac{\sin ax}{x}\right) dx = \int_0^\infty \cos ax dx$ is divergent.

Hence, formula (8.8) does not hold.

A simple sufficient condition for (8.8) to hold shall be given below. In what follows I, J will denote an interval $[a, b]$, or $[a, +\infty)$, or $(-\infty, b]$, or even $(-\infty, \infty)$. The symbol $\int_I f dx$ will denote the integral over the corresponding interval.

Theorem 8.4 *Assume $f(x, y)$ is a continuous function for $x \in I$ and $y \in J$, such that $f_y(x, y)$ is continuous and $|f_{yy}(x, y)| \le h(x)$, and $\int_I h dx$ is convergent.*
Then

$$\frac{d}{dy} \int_I f(x, y) dx = \int_I \frac{\partial f(x, y)}{\partial y} dx.$$

Note: When $I = [a, b]$ and $J = [\alpha, \beta]$, and f_{yy} is continuous for $x \in I$ and $y \in J$, the conditions of the theorem are satisfied and, and hence, the differentiation formula is valid.

Proof. By the Mean Value Theorem we have, for each $x \in I$,

$$f(x, y + k) = f(x, y) + k f_y(x, y) + \frac{1}{2} k^2 f_{yy}(x, y + \theta k), \quad \text{for some} \quad \theta, \ 0 < \theta < 1.$$

Thus

$$\left| \frac{f(x, y + k) - f(x, y)}{k} - f_y(x, y) \right| \le \frac{1}{2} \cdot k \cdot h(x).$$

Let

$$F(y) = \int_I f(x, y) dx.$$

To determine whether or not $F(y)$ is differentiable, consider the following:

$$\begin{aligned}
\left|\frac{F(x,y+k)-F(y)}{k}-\int_I f_y(x,y)dx\right| &= \left|\int_I\left[\frac{f(x,y+k)-f(x,y)}{k}-f_y(x,y)\right]dx\right| \\
&\le \int_I\left|\frac{f(x,y+k)-f(x,y)}{k}-f_y(x,y)\right|dx \\
&\le \frac{|k|}{2}\int_I h(x)dx \le M\,|k|,
\end{aligned}$$

for some constant M. Thus,

$$\lim_{k\to 0}\frac{F(y+k)-F(y)}{k}=\int_I f_y(x,y)dx,$$

as required. ∎

8.9 Evaluating Integrals Depending on a Parameter

We shall illustrate the ideas involved by means of the following examples.

Example 8.16 *Let*

$$\Phi(x)=\int_0^\infty t^{-1}e^{-t}(1-\cos xt)dt.$$

Determine $\Phi'(x)$. *Hence, or otherwise, evaluate* $\Phi(x)$.

Solution. We have

$$\Phi'(x)=\int_0^\infty e^{-t}\sin xtdt.$$

Now

$$\begin{aligned}
I=\int_0^\infty e^{-t}\sin xtdt &= \sin xt(-e^{-t})\Big|_0^\infty + x\int_0^\infty e^{-t}\cos xtdt \\
&= x\left[\cos xt(-e^{-t})\Big|_0^\infty - x\int_0^\infty e^{-t}\sin xtdt\right] \\
&= x(1-xI).
\end{aligned}$$

Hence $I=x-x^2I$, so that $I=\Phi'(x)=\dfrac{x}{x^2+1}$.
Thus

$$\Phi(x)=\int\frac{x}{x^2+1}dx+C=\frac{1}{2}\log(x^2+1)+C.$$

But $\Phi(0)=0$, so that $C=0$.
Thus

$$\Phi(x)=\frac{1}{2}\log(x^2+1).$$ ∎

Example 8.17

Let

$$G(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x} dx, \quad y > 0.$$

(i) Find $G'(y)$.

(ii) Hence evaluate $G(y)$.

(iii) Deduce that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

(iv) Show that

$$\int_0^\infty \frac{\sin ax}{x} dx = \begin{cases} \pi/2 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -\pi/2 & \text{if } a < 0. \end{cases}$$

Solution.

(i) For $y > 0$, we have

$$\begin{aligned} G'(y) &= \int_0^\infty \frac{\partial}{\partial y} \left[e^{-xy} \frac{\sin x}{x} \right] dx \\ &= -\int_0^\infty e^{-xy} \sin x dx \\ &= -\frac{1}{y} \int_0^\infty e^{-t} \sin\left(\frac{t}{y}\right) dt, \quad \text{by the substitution } t = xy, \\ &= -\frac{1}{y} \cdot \frac{1/y}{(1/y)^2 + 1}, \quad \text{by the previous Example 8.16.} \end{aligned}$$

Hence

$$G'(y) = -\frac{1}{y^2 + 1}, \quad y > 0.$$

This could also have been obtained by integrating by parts the integral

$$\int_0^\infty e^{-xy} \sin x dx.$$

(ii) Thus

$$G(y) = -\int \frac{1}{y^2 + 1} dy = -\arctan y + C.$$

Now

$$\lim_{y \to +\infty} G(y) = \lim_{y \to +\infty} \int_0^\infty e^{-xy} \frac{\sin x}{x} dx = 0.$$

Hence $0 = -\pi/2 + C$, so that $C = \pi/2$.

(iii) Hence

$$\int_0^\infty e^{-xy}\frac{\sin x}{x}dx = -\arctan y + \frac{\pi}{2},$$

valid for $y > 0$. Taking limits of both sides as $y \to 0+$, we get

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

(iv) Let $a > 0$ and $u = ax$. Then

$$\int_0^\infty \frac{\sin ax}{x}dx = \int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}.$$

For $a < 0$, put $b = -a$. Then

$$\int_0^\infty \frac{\sin ax}{x}dx = \int_0^\infty \frac{\sin(-bx)}{x}dx = -\int_0^\infty \frac{\sin bx}{x}dx = -\frac{\pi}{2}.$$

Clearly, if $a = 0$ then

$$\int_0^\infty \frac{\sin ax}{x}dx = 0. \quad \blacksquare$$

Note. Two important limits were considered above without proper justification. The justification is left as an exercise:

(i) $$\lim_{y\to\infty}\int_0^\infty e^{-xy}\frac{\sin x}{x}dx = 0;$$

(ii) $$\lim_{y\to 0+}\int_0^\infty e^{-xy}\cdot\frac{\sin x}{x}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

Example 8.18

Let

$$f(y) = \int_0^\infty \frac{\sin xy}{x(x^2+1)}dx, \quad y > 0.$$

(i) Show that $f''(y) = f(y) - \pi/2$.

(ii) Hence show that $f(y) = \pi(1 - e^{-y})/2$.

Solution.
(i) We have

$$f'(y) = \int_0^\infty \frac{\cos xy}{x^2+1}dx$$

$$f''(y) = -\int_0^\infty x\cdot\frac{\sin xy}{x^2+1}dx$$

$$= \quad -\int_0^\infty \left(\frac{x^2+1}{x} - \frac{1}{x}\right) \frac{\sin xy}{x^2+1} dx$$

$$= \quad -\int_0^\infty \frac{\sin xy}{x} dx + \int_0^\infty \frac{\sin xy}{x(x^2+1)} dx$$

$$= \quad -\frac{\pi}{2} + f(y), \quad \text{for } y > 0.$$

(ii) Solve the differential equation

$$f''(y) - f(y) = -\frac{\pi}{2},$$

to obtain

$$f(y) = Ae^y + Be^{-y} + \frac{\pi}{2},$$

where A, B are constants. Thus

$$f'(y) = Ae^y - Be^{-y}.$$

Now, $f(y) \to 0$, as $y \to 0+$, and

$$\lim_{y\to 0+} f'(y) = \int_0^\infty \frac{1}{x^2+1} dx = \frac{\pi}{2}.$$

[As in the previous example, these limits have to be carefully justified.]
Hence,

$$\begin{cases} 0 = A + B + \pi/2 \\ \pi/2 = A - B. \end{cases}$$

So that $A = 0$, $B = -\pi/2$. Hence

$$f(y) = \frac{\pi}{2}\left(1 - e^{-y}\right).$$

8.10 Exercises

8.1 Let

$$\Phi(a) = \int_0^\pi \log(1 - 2a\cos x + a^2) dx.$$

Show that $\Phi'(a) = 0$. Hence evaluate $\Phi(a)$ for $|a| < 1$ and $|a| > 1$.

8.2 Let

$$F(y) = \int_0^y \log(1 + \tan x \cdot \tan y) dx, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Show that $F'(y) = \log(\sec y) + y\tan y$. Hence, deduce that

$$F(y) = y \log \sec y.$$

8.3 Evaluate
$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx, \quad 0 < a < b.$$

8.4 Show that
$$\int_0^1 \left(\int_0^\infty \frac{1}{1 + x^2 + y^2} dx \right) dy = \int_0^\infty \left(\int_0^1 \frac{1}{1 + x^2 + y^2} dy \right) dx.$$
Deduce that
$$\int_0^{\pi/2} \frac{\arctan(\sin\theta)}{\sin\theta} d\theta = \frac{\pi}{2} \log(1 + \sqrt{2}).$$

8.5 Let
$$\Phi(y) = \int_0^\infty \left(\frac{\sin xy}{x} \right)^2 dx.$$
Show that $\Phi'(y) = \pi/2$ for $y > 0$. Hence evaluate $\Phi(y)$, for all y.

8.6 Let
$$\Phi(a) = \int_0^\infty e^{-(x^2+a^2)/x^2} dx.$$
Show that $\Phi'(a) = -2\Phi(a)$. Hence determine $\Phi(a)$.
Hint:
$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

8.7 Consider the Gamma function
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$
Justify the following, specifying for which values of x it is true:
$$\Gamma'(x) = (x - 1) \int_0^\infty t^{x-2} \cdot e^{-t} \cdot \ln t \cdot e^{-t} dt.$$

8.8 Assume $a > 0$. Which of the following integrals are convergent?

(i) $\displaystyle\int_a^\infty \frac{1}{x^{4/3}} dx,$ **(ii)** $\displaystyle\int_a^\infty \frac{1}{c^2 + x^2} dx,$

(iii) $\displaystyle\int_a^\infty \frac{x}{c^2 + x^2} dx,$ **(iv)** $\displaystyle\int_a^\infty \frac{x^2}{c^2 + x^2} dx.$

8.9 Show that
$$\int_a^\infty f(x) g'(x) dx = \left(\lim_{x\to+\infty} f(x) \cdot g(x) \right) - f(a) \cdot g(a) - \int_a^\infty f'(x) \cdot g(x) dx.$$
Hence, or otherwise, obtain the formula:
$$\int_1^\infty \frac{\sqrt{x}}{(1 + x)^2} dx = \frac{1}{2} + \frac{\pi}{4}.$$

Hint: Put $x = t^2, \quad t > 0.$

8.10 Show that the integral

$$\int_0^\infty \frac{x^{s-1}}{1+x}dx$$

is convergent if and only if $0 < s < 1$.

8.11 Show that

$$(\mathbf{i}) \qquad \lim_{y\to\infty}\int_0^\infty e^{-xy}\frac{\sin x}{x}dx = 0;$$

$$(\mathbf{ii}) \qquad \lim_{y\to 0+}\int_0^\infty e^{-xy}\cdot\frac{\sin x}{x}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

Chapter 9

Sequences of Functions and Power Series

9.1 Pointwise and Uniform Convergence

Definition 9.1 *Let* $\{f_n\}$ *be a sequence of functions defined on a given interval* I *and let* f *be a function that is also defined on* I. *If* $\lim_{n\to\infty} f_n(x) = f(x)$ *for each point* x *in* I, *we say that the sequence* $\{f_n\}$ **converges pointwise** *to the function* f *on* I. *Thus,* $\{f_n\}$ *converges pointwise to* f *on* I, *if*

$$\forall \varepsilon > 0 \ \ \forall x \in I \ \ \exists N_{\varepsilon,x} \quad (n > N_{\varepsilon,x} \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon)\,.$$

Definition 9.2 *Let* $\{f_n\}$ *be a sequence of functions defined on a given interval* I *and let* f *be a function that is also defined on* I. *The sequence* $\{f_n\}$ *is said to* **converge uniformly** *to* f *on* I *if*

$$\forall \varepsilon > 0 \ \ \exists N_{\varepsilon} \ \ \forall x \in I \quad (n > N_{\varepsilon} \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon)\,.$$

The concept of convergence in the sense of Definition 9.1 is called *pointwise* because it relates to the behaviour of the sequence $\{f_n(x)\}$ at each point x — The value of N in this case depends on the point $x \in I$, as well as on ε, in contrast to the uniform convergence defined by Definition 9.2, where the value for N does not depend on the choice of x. Note the order of the quantifiers $\forall x \in I$ and $\exists N$ in the two definitions. Uniform convergence has to do with the behaviour of $\{f_n\}$ over the whole interval.

When $\{f_n\}$ converges to f pointwise, then

1. it may happen that $f(x)$ is not continuous even if every $f_n(x)$ is continuous (see Example 9.4);

2. it does not follow that $f(x)$ is differentiable even if every $f_n(x)$ is differentiable (see Example 9.6); even if f is differentiable, it may not be true that $f'(x) = \lim_{n\to\infty} f'_n(x)$, (see Example 9.7);

3. it does not follow that
$$\int_I f(x)dx = \lim_{n\to\infty} \int_I f_n(x)dx,$$
even if the above limit exists (see Example 9.5).

$$f_n(x) = x + \frac{1}{n}\sin nx, \quad -\pi \le x \le \pi$$

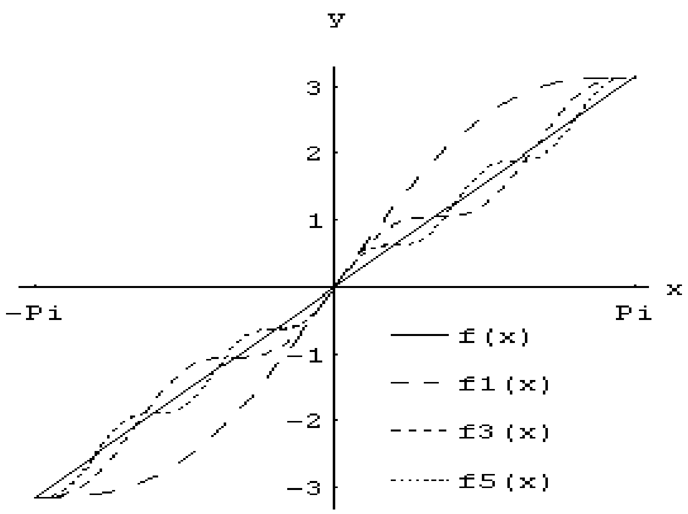


Figure 9.1: Illustrating uniform convergence of $\{f_n(x)\}$ to $f(x) = x$ on the interval $[-\pi,\ \pi]$.

Example 9.1 *Show that the sequence of functions* $\{f_n\}$*, where*
$$f_n(x) = x + \frac{1}{n}\sin nx, \quad n = 1, 2, \ldots$$
is uniformly convergent to the function $f(x) = x$ *in any interval* $I \subset \mathbb{R}$*.*

Solution. Let $\varepsilon > 0$ be given. We have
$$|f_n(x) - f(x)| = \frac{1}{n}|\sin nx| \le \frac{1}{n} < \varepsilon$$
for all values of x, provided $n > N = [\frac{1}{\varepsilon}]$.

Refer to Figure 9.1 to see the behaviour of selected functions of the sequence, namely $f_n(x)$ for $n = 1$, $n = 3$, and $n = 5$, over the interval $(-\pi,\ \pi)$. ∎

$$f_n(x) = \frac{1}{nx}, \quad 0.01 \le x \le 1$$

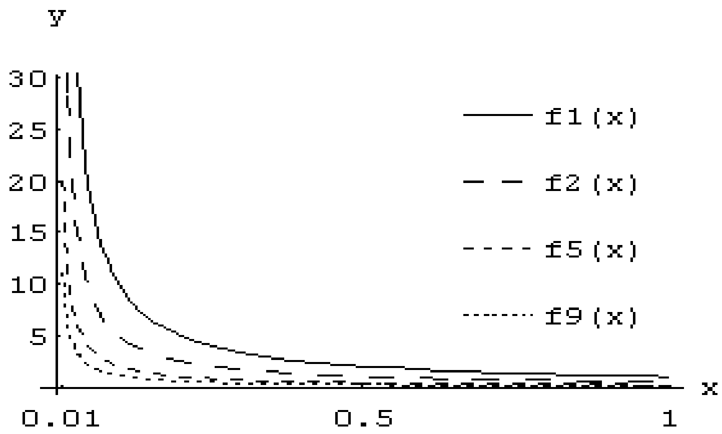


Figure 9.2: $\{f_n(x)\}$ converges uniformly to $f(x) = 0$ on the interval $[0.01,\ 1]$.

Example 9.2

Consider the sequence of functions $\{f_n\}$, where

$$f_n(x) = \frac{1}{nx} \quad n = 1, 2, \ldots.$$

(a) Show that $\{f_n(x)\}$ converges pointwise to $f(x) = 0$ in the interval $I = (0, +\infty)$, but not uniformly.

(b) Show that $\{f_n\}$ converges uniformly to $f(x) = 0$ on any interval (c, ∞), where $c > 0$.

Solution.

(a) Let x be any fixed point of the interval $(0, \infty)$. Then $\lim\limits_{n\to\infty} \frac{1}{nx} = 0$. Thus $\{f_n(x)\}$ converges pointwise to the function $f(x) = 0,\ \ 0 < x < \infty$.

This convergence is not uniform, since we cannot find such N, independent of x, that for every $\varepsilon > 0$ the inequality

$$|f_n(x) - f(x)| = \frac{1}{nx} < \varepsilon$$

holds for all values of x, $0 < x < \infty$, when $n > N$.

(b) If $I = (c, \infty)$, then

$$\forall \varepsilon > 0 \ \exists N = \left[\frac{1}{c\varepsilon}\right] \ \forall x \quad \left(c < x < \infty \ \&\ n > N \quad \Longrightarrow \quad \left|\frac{1}{nx} - 0\right| = \frac{1}{nx} < \frac{1}{Nc} < \varepsilon\right).$$

Figure 9.2 shows us graphs of selected functions $f_n(x)$ for $x \in [0.01,\ 1]$. ∎

$$f_n(x) = \frac{nx}{e^{nx^2}}, \ \ 0 \le x \le 1$$

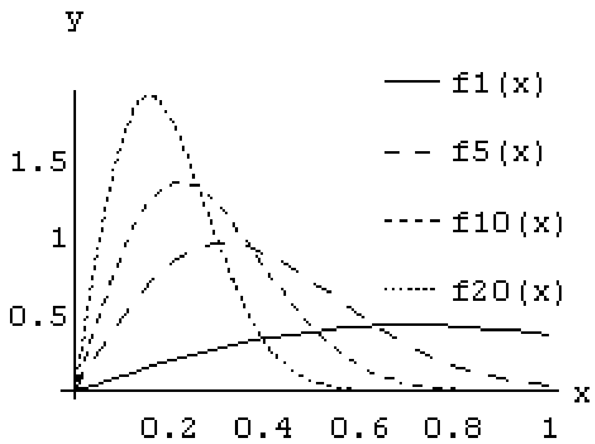


Figure 9.3: Selected functions $f_n(x)$, $(n = 1, 5, 10, 20)$ of a sequence that converges pointwise (but not uniformly) to the function $f(x) = 0$, $x \in [0, 1]$.

Example 9.3 *Show that the sequence given by* $f_n(x) = nxe^{-nx^2}$, $n = 1, 2, \ldots,$ *converges pointwise, but not uniformly, to the function* $f(x) = 0$ *on the interval* $I = [0, 1]$ *(Refer to Figure 9.3).*

Solution. Let x be any real number. Then the sequence $\{f_n(x)\}$ converges to the limit 0:

$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{t\to\infty} \frac{tx}{e^{tx^2}} = \lim_{t\to\infty} \frac{x}{e^{tx^2} \cdot x^2} = 0,$$

for $x \neq 0$. If $x = 0$, clearly, $\lim_{n\to\infty} f_n(x) = 0$.

Hence $\{f_n\}$ converges pointwise to f on the whole real line, $-\infty < x < \infty$, and in particular on the interval I.

Suppose now that $\{f_n\}$ converges uniformly to f on the interval $I = [0, 1]$. Then given any $\varepsilon > 0$, the inequality $|f_n(x)| < \varepsilon$ must hold for all $x \in [0, 1]$ and all $n > N$, where N does not depend on x. Suppose that such N exists and let

$$x = \frac{1}{\sqrt{2n}} \in I, \ \ n = 1, 2, \ldots. \tag{9.1}$$

Then $f_n(x) = \dfrac{\sqrt{n}}{\sqrt{2e}} \to +\infty$, as $n \to \infty$. We see that $|f_n(x)| < \varepsilon$ cannot hold for all x values defined by (9.1) and all $n > N$, since $\{\dfrac{\sqrt{n}}{\sqrt{2e}}\}$ is unbounded, as $n \to \infty$. ∎

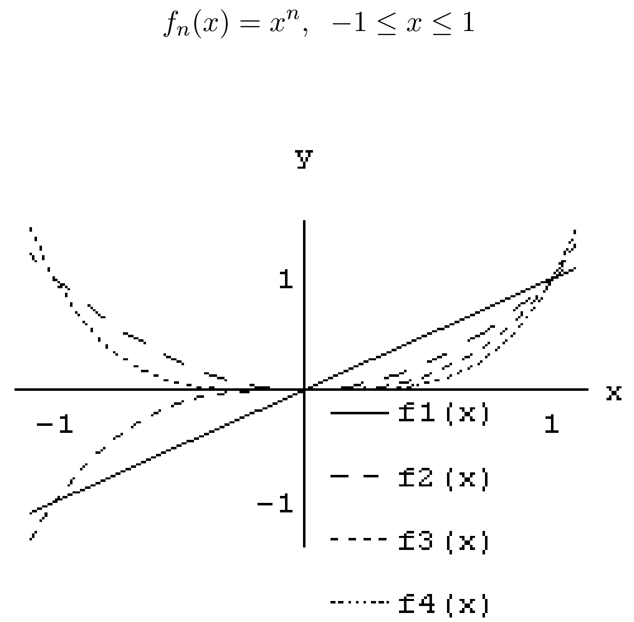


Figure 9.4: Illustrating convergence of the sequence $\{f_n(x)\}$ to 0 when $|x| < 1$ and to 1 when $x = 1$.

Example 9.4 *Examining convergence of $f_n(x) = x^n$ on the interval $I = (-1, 1]$.*

(a) We have

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, & |x| < 1 \\ 1, & x = 1 \end{cases}$$

Thus $\{f_n\}$ converges pointwise to f on the interval I.

(b) $\{f_n\}$ does not converge uniformly to f on $I = (-1, +1]$, since, if $x = 1 - \dfrac{1}{n}$, then

$$f_n(x) = (1 - \frac{1}{n})^n \to \frac{1}{e}.$$

(c) $\{f_n\}$ converges uniformly to f on any interval of the form $[-a, a]$, where $a < 1$, since $|x^n - 0| = |x^n| \le a^n < \varepsilon$ for all $x \in [-a, a]$ and all $n > N = [\frac{\ln \varepsilon}{\ln a}]$. ∎

Theorem 9.1 Uniform Convergence and Continuity
Suppose that $\{f_n\}$ is a sequence of continuous functions defined on a closed interval $I = [a,b]$. If f_n converges uniformly on I to a function f then f is continuous on I.

Proof. Let $x_o \in I$ and $\varepsilon > 0$ be given. Consider the identity

$$f(x) - f(x_o) = f(x) - f_n(x) + f_n(x) - f_n(x_o) + f_n(x_o) - f(x_o).$$

Hence, for any $x \in I$ and any $n = 1, 2, \ldots$, we get

$$|f(x) - f(x_o)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_o)| + |f_n(x_o) - f(x_o)|. \tag{9.2}$$

Since $f_n \to f$, uniformly on I, there is an integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in I$ and $n \ge N$. Thus

$$|f(x) - f(x_o)| \le \frac{\varepsilon}{3} + |f_n(x) - f_n(x_o)| + \frac{\varepsilon}{3}, \quad n \ge N, \quad x \in I.$$

The above inequality holds for any value of n such that $n \ge N$; in particular for $n = N$:

$$|f(x) - f(x_o)| \le \frac{\varepsilon}{3} + |f_N(x) - f_N(x_o)| + \frac{\varepsilon}{3}, \quad x \in I. \tag{9.3}$$

Now, the function $f_N(x)$ is continuous on I, so that there exists a $\delta > 0$ such that

$$|x - x_o| < \delta \quad \Longrightarrow \quad |f_N(x) - f_N(x_o)| < \frac{\varepsilon}{3}.$$

Therefore, by (9.3), $|f(x) - f(x_o)| \le \varepsilon$ provided $|x - x_o| < \delta$. This proves that f is continuous at any point x_o arbitrarily chosen in the interval I. Hence, f is continuous on I. ∎

Example 9.5 *Give an example of a sequence $\{f_n(x)\}$ that converges (pointwise) on a given interval to a integrable function, but*

$$\lim_{n\to\infty} \int_I f_n(x)dx \ne \int_I \lim_{n\to\infty} f_n(x)dx.$$

Solution. Consider the following sequence of functions:

$$f_n(x) = \begin{cases} 2n^2 x, & 0 \le x \le 1/2n \\ 2n - 2n^2 x, & 1/2n \le x \le 1/n \\ 0, & 1/n \le x \le 1. \end{cases}$$

Selected functions of the sequence $\{f_n(x)\}$ are shown in Figure 9.5. We note that $\{f_n\}$ converges pointwise to $f(x) = 0$ on the interval $I = [0,1]$. It can be shown that $\{f_n(x)\}$ does not converge uniformly on I.

We have

$$\int_0^1 f_n(x)dx = \int_0^{1/2n} 2n^2x^2dx + \int_{1/2n}^{1/n} (2n - 2n^2x)dx = \frac{1}{2},$$

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le 1/2n \\ 2n - 2n^2x, & 1/2n \le x \le 1/n \\ 0, & 1/n \le x \le 1. \end{cases}$$

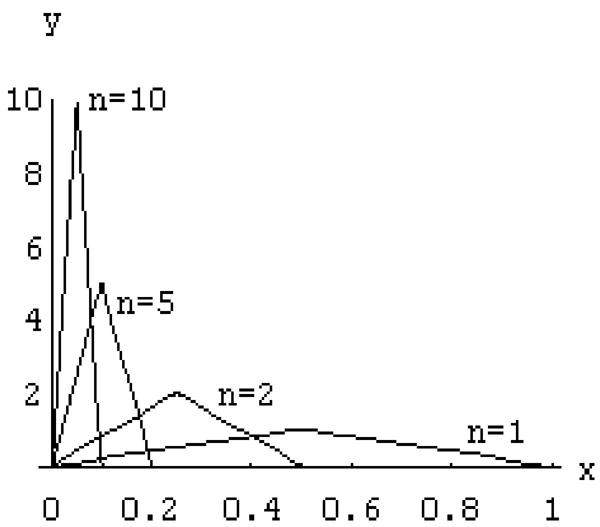


Figure 9.5: An example of a sequence $\{f_n\}$ that converges to a integrable function f on the interval $I = [0, 1]$, but $\int_I f(x)dx \neq \lim_{n\to\infty} \int_I f_n(x)dx$.

but

$$\int_0^1 f(x)dx = \int_0^1 odx = 0.$$

Hence,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx. \quad \blacksquare$$

Theorem 9.2 Uniform Convergence and Integration
Suppose that $\{f_n\}$ is a sequence of functions defined on a closed interval $I = [a, b]$. If f_n converges uniformly on I to a function f, then f is integrable on I and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx. \tag{9.4}$$

Proof. To show that f is integrable we shall prove that, for a given $\varepsilon > 0$, there is a partition $\mathcal{P}$ for which

$$|\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P} < f)| < \varepsilon.$$

Fix n such that
$$\forall x \in I \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}.$$
Since f_n is integrable, there exists $\mathcal{P}$ for which
$$|\mathcal{U}(\mathcal{P}, f_n) - \mathcal{L}(\mathcal{P}, f_n)| < \frac{\varepsilon}{3}.$$
For that partition $\mathcal{P}$, we have
$$\begin{aligned} |\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f)| &\leq |\mathcal{U}(\mathcal{P}, f) - \mathcal{U}(\mathcal{P}, f_n)| \\ &\quad + |\mathcal{U}(\mathcal{P}, f_n) - \mathcal{L}(\mathcal{P}, f_n)| + |\mathcal{L}(\mathcal{P}, f_n) - \mathcal{L}(\mathcal{P}, f)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$
since, if h and k are any two functions for which $|h(x) - k(x)| < \varepsilon'$ for all x in I, and $\mathcal{P}'$ is <u>any</u> partition, then
$$|\mathcal{U}(\mathcal{P}', h) - \mathcal{U}(\mathcal{P}', k)| \leq \varepsilon'(b-a) \quad \text{and} \quad |\mathcal{L}(\mathcal{P}', h) - \mathcal{L}(\mathcal{P}', k)| \leq \varepsilon'(b-a).$$

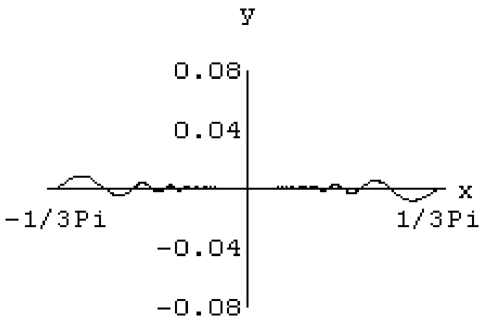
Now we shall show that (9.4) holds. Let $\varepsilon > 0$ be given. Since f_n converges uniformly to f on the interval I, there is some N such that
$$\forall x \in I \quad (n > N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}).$$

We have already proved that f is integrable. Thus for $n > N$, we have
$$\begin{aligned} \left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| &= \left| \int_a^b [f_n(x) - f(x)]dx \right| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &\leq \int_a^b \frac{\varepsilon}{b-a} dx = \varepsilon. \end{aligned}$$
We conclude that
$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx. \quad \blacksquare$$

The behaviour of uniform convergence with respect to differentiation is more complex. In fact, a sequence of differentiable functions $\{f_n(x)\}$ may converge uniformly to a function f which fails to be differentiable. Even if f is differentiable, it may not be true that
$$f'(x) = \lim_{n\to\infty} f_n'(x).$$
Two examples that illustrate the phenomena are given below.

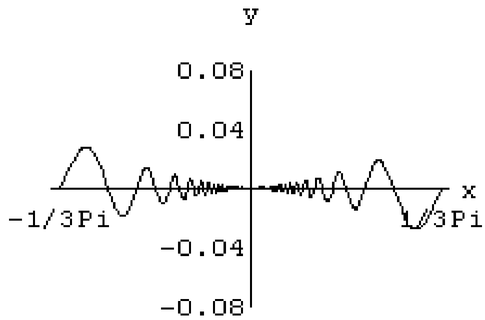
(a)

$$f_1(x) = \begin{cases} |x|^2 \sin 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



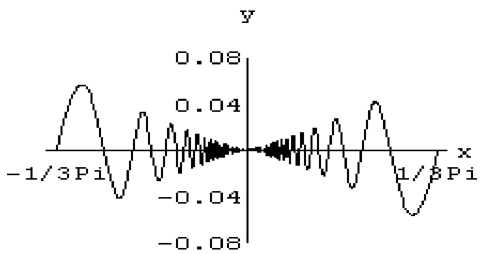
(b)

$$f_5(x) = \begin{cases} |x|^{6/5} \sin 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



(c)

$$f_{10}(x) = \begin{cases} |x|^{11/10} \sin 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



(d)

$$f(x) = \begin{cases} |x| \sin 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

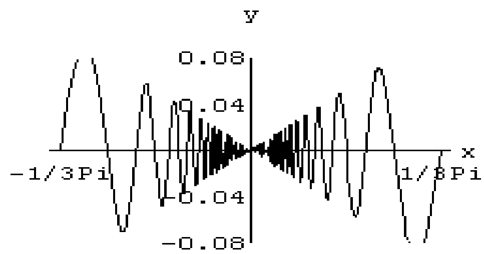


Figure 9.6: A sequence of differentiable functions converging to a function that is not differentiable.

Example 9.6

Let

$$f_n(x) = \begin{cases} |x|^{1+1/n} \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\{f_n\}$ converges to

$$f(x) = \begin{cases} |x| \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Note that each f_n is differentiable on $\mathbb{R}$, but f is not differentiable at $x = 0$. ∎

$$f_n(x) = \frac{1}{n}\sin n^2 x, \quad 0 \le x \le \pi$$

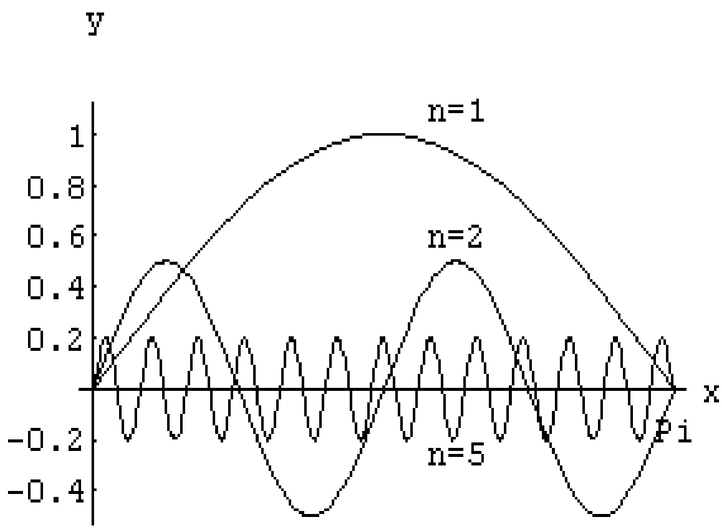


Figure 9.7: An example of a sequence $\{f_n\}$ that converges uniformly to a differentiable function f, but $f'(x) \neq \lim\limits_{n\to\infty} f_n'(x)$.

Example 9.7

Let $\{f_n\}$ be given by $f_n(x) = \frac{1}{n}\sin(n^2 x)$, $\quad n = 1, 2, \ldots$. Clearly, the sequence converges uniformly to $f(x) = 0$ on the whole real line, $-\infty < x < \infty$, since

$$|f_n(x)| = \left|\frac{1}{n}\sin(n^2 x)\right| \le \frac{1}{n}, \quad n = 1, 2, \ldots, \quad -\infty < x < \infty.$$

We have

$$f_n'(x) = n\cos(n^2 x),$$

and we can see that

$$\lim_{n\to\infty} f_n'(x) \neq f'(x) = 0,$$

as $\lim\limits_{n\to\infty} n\cos(n^2 x)$ does not always exist (for example, it does not exist if $x = 0$). ∎

The following result is useful and provides conditions under which the uniform limit of differentiable functions is differentiable.

Theorem 9.3 Uniform Convergence and Differentiation
Suppose that $\{f_n\}$ *is a sequence of functions defined on* $I = [a, b]$ *that are differentiable on* I*. Suppose further that for some functions* f *and* g *defined on* I*, where* g *is continuous, we have*

(i) $f_n \to f$ *in* I*,*

(ii) $f_n' \to g$ *uniformly in* I*.*

Then f *is a continuous and differentiable function on* I*, and we have*

$$g(x) = f'(x) = \lim_{n\to\infty} f_n'(x) \text{ for } x \in I.$$

Proof. Let x be any number of the interval I. Applying Theorem 9.2 to the interval $[a, x]$, we get

$$\int_a^x g(t)dt = \lim_{n\to\infty} \int_a^x f_n'(t)dt = \lim_{n\to\infty} [f_n(x) - f_n(a)] = f(x) - f(a),$$

so that

$$\int_a^x g(t)dt = f(x) - f(a).$$

Differentiating both sides of the above equation gives

$$\frac{d}{dx} \int_a^x g(t)dt = f'(x) - 0,$$

and we obtain

$$g(x) = f'(x), \quad x \in I = [a, b]. \quad \blacksquare$$

9.2 Power Series

Definition 9.3 *A* **power series** *about a point* x_o *is*

$$\sum_{n=0}^{\infty} a_n(x - x_o)^n = a_o + a_1(x - x_o) + a_2(x - x_o)^2 + \cdots, \tag{9.5}$$

where a_n*,* $n = 0, 1, 2, \ldots$*, are constant coefficients. In particular, when* $x_o = 0$*, the power series* (9.5) *takes the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + \cdots. \tag{9.6}$$

Evidently, every power series (9.6) is convergent at the point $x = 0$.

A power series $\sum_{n=0}^{\infty} a_n x^n$ has one of the following properties:

1. the series is divergent for all values of x, except $x = 0$;
2. the series is convergent for all values of x, $-\infty < x < \infty$;
3. there is a number $R > 0$, called the radius of convergence of the series, such that $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$ and is divergent for $|x| > R$.

A series that converges for all x, $-\infty < x < \infty$, is said to have radius of convergence $R = \infty$ and the series that only converges for $x = 0$ is said to have radius of convergence $R = 0$.

The following theorem gives a formula for the radius of convergence R.

Theorem 9.4 *Consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n. \tag{9.7}$$

Let

$$\alpha = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}. \tag{9.8}$$

Then the radius of convergence of the power series (9.7) is

$$R = \begin{cases} 0 & \text{if } \alpha = \infty \\ \infty & \text{if } \alpha = 0 \\ \dfrac{1}{\alpha} & \text{otherwise.} \end{cases}$$

Proof.

Case 1. Suppose that α is finite and $\alpha \neq 0$. Consider any x with $|x| < \frac{1}{\alpha}$. There is a $\delta > 0$ such that

$$|x| < \frac{1/\alpha}{1+\delta} < \frac{1}{\alpha}. \tag{9.9}$$

Let $\varepsilon = \alpha\delta/2$. We conclude that there is an integer N such that

$$n > N \implies |a_n|^{1/n} < \alpha + \varepsilon.$$

Hence

$$|a_n|^{1/n} < \alpha\left(1 + \frac{\delta}{2}\right) \implies |a_n| < \alpha^n \left(1 + \frac{\delta}{2}\right)^n, \text{ for } n > N.$$

Using (9.9), we get, for $n > N$,

$$|a_n x^n| < \alpha^n \left(1 + \frac{\delta}{2}\right)^n \frac{1}{\alpha^n (1+\delta)^n} = \left(\frac{1+\delta/2}{1+\delta}\right)^n = \beta^n,$$

where $0 < \beta < 1$. Since $|\beta| < 1$, the series $\sum \beta^n$ converges and by the Comparison Test, the series $\sum |a_n x^n|$ also converges.

We have shown that the power series (9.7) converges for any x with $|x| < \frac{1}{\alpha} = R$.

Also, if $|x| > 1/\alpha$, then $\alpha > 1/|x|$, so that, by definition of α, there are an infinity of values of n for which

$$|a_n|^{1/n} > \frac{1}{|x|}.$$

For such n we have $|a_n x^n| > 1$. Hence $\lim_{n\to\infty} a_n x^n \neq 0$, so that $\sum a_n x^n$ diverges.

Thus R is the radius of convergence.

Case 2. $\alpha = 0$.

If $\alpha = 0$, then $\lim_{n\to\infty} |a_N|^{1/n} = 0$. Given any $x \neq 0$, we have

$$|a_n|^{1/n}|x| \leq r < 1, \quad n \geq N,$$

for some N and some r. Then $|a_n x^n| \leq r^n$. But $\sum r^n$ converges, hence so does $\sum a_n x^n$. Hence the series converges for all x, so that $R = \infty$.

Case 3. $\alpha = +\infty$.

If

$$\limsup_{n\to\infty} |a_n|^{1/n} = +\infty,$$

then for $x \neq 0$, the sequence $\{|a_n||x^n|\}$ does not tend to 0. Hence, the series $\sum a_n x^n$ diverges. Thus $R = 0$. ∎

9.3 Taylor and Maclaurin Series

Consider a function $f(x)$ in a neighbourhood $N(x_o)$ of the point $x = x_o$. Suppose that f and its derivatives $f^{(k)}(x)$, for $k = 1, 2, \ldots n$, are continuous in $N(x_o)$. Then f can be approximated by the n-th degree Taylor polynomial

$$P_{n,\,x_o}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_o)}{k!}(x - x_o)^k.$$

Formally, by Taylor's theorem,

$$f(x) = P_{n,\,x_o}(x) + R_n(x),$$

where the remainder $R_n(x)$ may be specified in different forms.

Suppose now that $f^{(n)}(x_o)$ exists for all $n = 1, 2, \ldots$. We can then form a power series $\sum a_n (x - x_o)^n$, whose coefficients are the Taylor coefficients

$$a_n = \frac{f^{(n)}(x_o)}{n!}.$$

The question that naturally arises is under what conditions $f(x)$ can be represented as the corresponding power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n. \tag{9.10}$$

which is called the Taylor series of f at x_o. Note that, if the Taylor series (9.10) converges for $|x-x_o|<R,\;\; R>0$, it is not necessarily true that it converges to $f(x)$.

A necessary and sufficient condition for the power series (9.10) to converge to $f(x)$ is

$$\lim_{n\to\infty} R_n(x) = 0,$$

where

$$R_n(x) = f(x) - P_{n,\, x_o}(x).$$

If $\lim\limits_{n\to\infty} R_n(x) = 0$ for $|x-x_o|<R,\;\; R>0$, then we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n, \quad \text{for} \quad |x-x_o|<R,\;\; R>0,$$

and the power series (9.10) is called the **Taylor expansion** of f about the point x_o. The special case, $x_o=0$, in (9.10) is called the **Maclaurin expansion** of f.

The Taylor theorem (Theorem 6.1) gives the remainder $R_n(x)$ in the following form:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_o)^{n+1}, \quad 0<\xi<x.$$

Suppose that all the derivatives $f^{(k)}(x)$ are bounded in $|x-x_o|<R,\;\; R>0$, by the same constant M:

$$|f^{(k)}(x)| \le M, \;\; |x-x_o|<R. \tag{9.11}$$

Then, for any fixed value of x, $|x-x_o|<R$, we have

$$|R_n(x)| \le M\frac{(x-x_o)^{n+1}}{(n+1)!} \to 0, \quad \text{as} \quad n\to 0.$$

Hence, $f(x)$ can be represented by its Taylor expansion.

Definition 9.4 *A function f is said to be* **analytic** *at x_o if and only if f can be represented by its Taylor series in some neighbourhood of x_o.*

We can see that any function that satisfies (9.11) is analytic.

Example 9.8 *Showing that* $\sin x$ *and* $\cos x$ *are analytic at any point* $x \in \mathbb{R}$*.*

(a) If $f(x) = \sin x$, then $f^{(k)}(x) = \sin(x + k\pi/2)$, $k = 0, 1, 2, \ldots$, so that

$$|f^{(k)}(x)| \le M = 1, \quad -\infty < x < \infty.$$

Hence $R_n(x) \to 0$, as $n \to \infty$. Thus $f(x) = \sin x$ is analytic at any point $x \in \mathbb{R}$. Using the results of Example 6.3, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad -\infty < x < \infty. \tag{9.12}$$

(b) Let $f(x) = \cos x$. Following the logic of **(a)** and using the results of Example 6.4, we conclude that $f(x) = \cos x$ is analytic at any point $x \in \mathbb{R}$ and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad -\infty < x < \infty. \quad \blacksquare \tag{9.13}$$

Example 9.9 *Show that* $f(x) = e^x$ *is analytic at* x, $-\infty < x < \infty$.

Solution. If $f(x) = e^x$, then $f^{(k)}(x) = e^x$, $k = 0, 1, 2, \ldots$. For a fixed value of x, we have

$$|R_n(x)| \le \left| \frac{e^x}{(n+1)!} x^{n+1} \right| \le \frac{e^{|x|}|x|^{n+1}}{(n+1)!} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, $f(x) = e^x$ is analytic at any point $x \in \mathbb{R}$. Using the results of Example 6.2, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty. \quad \blacksquare$$

Now we give an example of a function that has all derivatives $f^{(n)}(x)$ in a neighbourhood of $x_o = 0$, ($|x| < \delta$, $\delta > 0$), and yet does not have a Taylor expansion valid for that interval.

Example 9.10 *A function that is not analytic at the point* 0 *and has derivatives* $f^{(n)}(0)$ *for all* $n \ge 1$.

Let

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

We evaluate the derivatives $f^{(n)}(0)$, $n = 1, 2, \ldots$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x},$$

$$\lim_{x \to 0-} \frac{f(x)}{x} = 0, \quad \text{since} \quad f(x) \equiv 0 \quad \text{for} \quad x < 0,$$

$$\lim_{x \to 0+} \frac{f(x)}{x} = \lim_{x \to 0+} \frac{1}{x} e^{-1/x} = \lim_{t \to \infty} t e^{-t} = \lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0.$$

Hence $f'(0) = 0$. Similarly,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0} \frac{f'(x)}{x}.$$

The limit from the left is obviously 0. The limit from the right is

$$\lim_{x\to 0+} \frac{f'(x)}{x} = \lim_{x\to 0+} x^{-3}e^{-1/x} = \lim_{t\to\infty} t^3 e^{-t} = \lim_{t\to\infty} \frac{t^3}{e^t} = \lim_{t\to\infty} \frac{3t^2}{e^t} = \lim_{t\to\infty} \frac{6t}{e^t} = 0.$$

Hence $f''(0) = 0$. By induction, we get $f^{(n)}(0) = 0$ for all $n = 1, 2, \ldots$. Hence the Taylor coefficients at 0 are

$$a_n = \frac{f^{(n)}(x_o)}{n!} = 0, \quad n = 1, 2, \ldots.$$

If we had a Taylor expansion of $f(x)$ at $x_o = 0$, then we would have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \equiv 0, \quad |x| < \delta,$$

which does not hold. ∎

9.4 Differentiation of Power Series

Consider a power series

$$\sum_{k=0}^{\infty} a_k (x - x_o)^k \tag{9.14}$$

with a given radius of convergence R, $R > 0$. For each x, $|x - x_o| < R$, the series (9.14) is a function of x:

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_o)^k.$$

We know that many familiar functions can be represented as power series, for example

$$f(x) = \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k, \quad \text{if} \quad |x| < 1.$$

In this section we deal with the question of whether a power series can be differentiated and how the derivative can be calculated.

Note, for example, that the sum of the series $\sum x^n$ is a differentiable function whose derivative is

$$f'(x) = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}, \quad |x| < 1.$$

Hence

$$\left(\sum_{n=0}^{\infty} x^n\right)' = \frac{1}{(1-x)^2}, \quad |x| < 1. \tag{9.15}$$

Theorem 9.5 Differentiation of Power Series
Let the power series $\sum a_n x^n$ have the radius of convergence $R > 0$ and let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < R.$$

Then $f(x)$ is differentiable for $|x| < R$ and

$$f'(x) = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, the series $\sum n a_n x^{n-1}$ has the radius of convergence R.

Note that an application of this theorem to (9.15) gives us a familiar result

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad |x| < 1.$$

Proof. Assuming that the radius of convergence of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is R, we are to prove that

(i) the series

$$\sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{9.16}$$

has radius of convergence R;

(ii) $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad |x| < R.$

Proof of (i) Let R' be the radius of convergence of (9.16). Since

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n, \quad x \neq 0, \quad \text{and} \quad \lim_{n \to \infty} \sqrt[n]{n} = 1,$$

we have

$$\limsup_{n \to \infty} \sqrt[n]{|n a_n|} = \lim_{n \to \infty} \sqrt[n]{n} \cdot \limsup_{n \to \infty} \sqrt[n]{|a_n|} = \limsup_{n \to \infty} \sqrt[n]{|a_n|},$$

which implies that $R' = R$.

Proof of (ii) Let c be any point of $I = (-R, R)$ and let r be another point of I such that $|c| < r < R$. Let h be such that $0 < |h| < r - |c|$, so that $c + h \in (-r, r)$.

To prove that

$$f'(c) = \sum_{n=1}^{\infty} n a_n c^{n-1}$$

we introduce D_h, a function of h defined as

$$D_h = \frac{f(c+h) - f(c)}{h} - \sum_{n=1}^{\infty} n a_n c^{n-1}$$

and prove that $D_h \to 0$, as $h \to 0$.

By the hypothesis,

$$f(c+h) = \sum_{n=0}^{\infty} a_n (c+h)^n, \quad f(c) = \sum_{n=0}^{\infty} a_n c^n,$$

so that

$$f(c+h) - f(c) = \sum_{n=0}^{\infty} a_n \left[(c+h)^n - c^n\right].$$

By Taylor's theorem applied to the function $g(x) = x^n$ we have

$$(c+h)^n = c^n + nhc^{n-1} + \frac{1}{2} n(n-1) h^2 (c + \theta h)^{n-2},$$

for some θ, $0 < \theta < 1$.

Hence

$$\begin{aligned} D_h &= \left| \frac{f(c+h) - f(c)}{h} - \sum_{n=1}^{\infty} n a_n c^{n-1} \right| = \left| \frac{1}{2h} \sum_{n=1}^{\infty} n(n-1) h^2 (c+\theta h)^{n-2} a_n \right| \\ &\le \frac{1}{2} |h| \sum_{n=1}^{\infty} n(n-1)\ a_n\ |c + \theta h|^{n-2}. \end{aligned}$$

Now, $|c + \theta h| \le |c| + |h| < r$, so

$$|D_h| \le \frac{1}{2} |h| \sum_{n=1}^{\infty} n(n-1) \left| a_n x^{n-2} \right|, \quad \text{where} \quad |x| = |c + \theta h| < R.$$

Applying **(i)** to the series $\sum n a_n x^{n-1}$ that has radius of convergence R, we arrive at the conclusion that the series

$$\sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} \tag{9.17}$$

has also radius of convergence R. Therefore (9.17) converges absolutely and consequently is bounded for $x = c + \theta h$, say:

$$\sum_{n=1}^{\infty} n(n-1) |a_n (c + \theta h)^{n-2}| \le M, \quad \text{for some} \quad M > 0, \quad \text{independent of } h.$$

Hence

$$|D_h| \le \frac{1}{2} |h| M \to 0, \quad \text{as} \quad h \to 0. \ \blacksquare$$

Example 9.11 *Obtain the Maclaurin expansion for* $\cos x$ *by application of Theorem 9.5 to the Maclaurin expansion of* $\sin x$*.*

Solution. By Example 9.8 **(a)**, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad -\infty < x < \infty.$$

By Theorem 9.5, we can differentiate the above series term by term to obtain:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n},$$

(compare with Example 9.8 (b)). ∎

Example 9.12

Prove that the power series

$$\sum_{n=1}^{\infty} n^2 x^n$$

converges for $|x| < 1$ and determine its sum. Hence evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Solution. If R denotes the radius of convergence of $\sum n^2 x^n$ then we have

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Hence $R = 1$ and the series converges for $|x| < R = 1$.

Consider now the series $\sum x^n$, with radius of convergence 1:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

On an application of Theorem 9.5 (twice), we obtain the formulas:

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad |x| < 1,$$

$$\sum_{n=1}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}, \quad |x| < 1.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=1}^{\infty}[n(n-1)+n]x^n \\ &= x^2\sum_{n=1}^{\infty} n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} nx^{n-1} \\ &= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\ &= \frac{x(1+x)}{(1-x)^3}, \quad |x| < 1.\end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \quad |x| < 1.$$

In particular, putting $x = 1/2$, we get

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6.$$

9.5 Integration of Power Series

Theorem 9.6 Integration of Power Series
Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

have radius of convergence R. Then, for any x, $|x| < R$, we have

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \tag{9.18}$$

and the power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \tag{9.19}$$

also has radius of convergence R.

Proof.
Let $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. The series for $f(x)$ and $g(x)$ have the same radius of convergence, since $g'(x) = f(x)$, as proved in Theorem 9.5. But then $\int_0^x f(t)dt = g(x) - g(0)$, by the Fundamental Theorem of Integral Calculus. Now $g(0) = 0$, hence (9.18) holds. ∎

Example 9.13 *Integrating power series term by term.*

(a) By Example 2.20, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1. \tag{9.20}$$

Let $z = -x$, where $|x| < 1$. Then we obtain

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1.$$

If $|x| < 1$ then, by Theorem 9.5, we can integrate the above series term by term:

$$\int_0^x \frac{1}{1+t} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt, \quad |x| < 1.$$

Hence we arrive at the following expansion for of function $\log(1+x)$:

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1. \tag{9.21}$$

(b) Let $z = -x^2$, where $|x| < 1$. Then 9.20 gives

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1.$$

Applying Theorem 9.5 to the above, we arrive at the following expansion of the function $\arctan x$:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad |x| < 1. \quad \blacksquare$$

Comments: Recall that in section 6.1 we dealt with the approximation of the function $\log(1+x)$ by its Taylor polynomials $P_{n,0}(x)$. In Figure 6.4 we illustrated this approximation on the interval $(-1, 1)$. The question that naturally arises is whether Taylor's polynomials can be used to approximate $f(x) = \log(1+x)$ on a wider interval, say for $x \in (-1, 2)$.

Example 9.13 ensures that the approximation is valid for $|x| < 1$. Now, for $|x| > 1$, the power series 9.21 diverges. Hence the polynomials $P_{n,0}(x)$, being the partial sums of the series 9.21, cannot be expected to provide an approximation of $f(x)$.

Refer to Figure 9.8 to see the behaviour of the polynomials $P_{n,0}(x)$ for $1 < x < 2$.

$$P_{n,0}(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k}$$

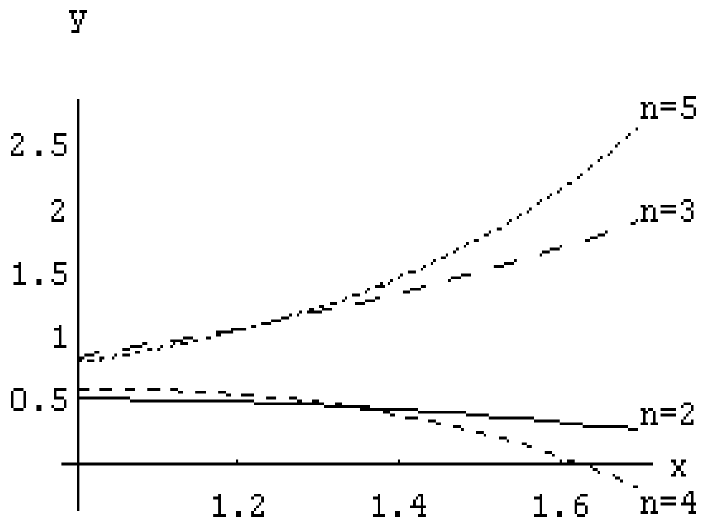


Figure 9.8: Illustrating divergence of the Taylor series of $f(x) = \log(1+x)$ for $x > 1$.

9.6 Exercises

9.1 Let

$$f_n(x) = 1 - x^2 + x^4 - \ldots + (-1)^n x^{2n}, \quad -1 < x < 1.$$

(a) Determine, for each x in $(-1, 1)$,

$$f(x) = \lim_{n\to\infty} f_n(x);$$

(b) Does $\{f_n\}$ converge uniformly to f? Consider the cases:

(i) $-1 < x < 1$,

(ii) $-1 + \delta < x < 1 - \delta$, where $0 < \delta < 1$.

9.2 If $|f_n(x) - f(x)| \le a_n$, for $n \ge 1$ and $x \in I$, show that $\{f_n\}$ converges uniformly to f on I, whenever $a_n \to 0$, as $n \to \infty$.

Use this result to justify the following formulas, where $-1 < x < 1$:

$$\log \frac{1}{1-x} = \int_0^x \frac{1}{1-t} dt = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots,$$

$$\frac{1}{2} \log \left(\frac{1+x}{1-x} \right) = \int_0^x \frac{1}{1-t^2} dt = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots.$$

9.3 Let $f_n(x) = x^n,\ \ 0 \le x \le 1$. Does $\{f_n\}$ converge uniformly? Justify your answer.

9.4 Let $f(x) = \sum(a_n x^n),\ \ |x| < R$ and let $g(x) = \sum(b_n x^n),\ \ |x| < R'$, where $R,\ R' > 0$. If $f(x) = g(x)$ for all x such that $|x| < c$, for some $c > 0$, show that $a_n = b_n$ for all n.

Thus, two power series that represent the same function must be identical.

9.5 Prove the following theorem.

Theorem 9.7 *Let R be the radius of convergence of $\sum(a_n x^n)$ and let K be a closed and bounded subset of the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.*

9.6 Let

$$y = \sum_{n=0}^{\infty} a_n x^n,\ \ |x| < R,$$

be a solution of the differential equation $y'' + y = 0$. Determine the coefficients a_n, and identify the solutions.

9.7 Obtain an explicit formula for a_n and b_n, given by:

$$(i) \qquad a_{n+2} = a_{n+1} - a_n; \quad a_o = 0,\ \ a_1 = 1.$$

$$(ii) \qquad b_{n+2} = b_{n+1} + b_n; \quad b_o = 0,\ \ b_1 = 1.$$

9.8 Prove the following theorem.

Theorem 9.8 *The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.*

9.9 Use the Fundamental Theorem of Integral Calculus and the above results showing that a power series may be integrated term by term within its domain of convergence, to justify that a power series may be differentiated term-by-term within its domain of convergence.

Index