LECTURE NOTES
IN PARTIAL DIFFERENTIAL EQUATIONS

Third Edition

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PREFACE

These lecture notes are designed for undergraduate students as a complementary reading text to an introductory course on Partial Differential Equations. It is assumed that the students have basic knowledge in Real Analysis.

The notes have been used for teaching the course MAT426 (PDE), Partial Differential Equations at the Faculty of Science, University of Botswana.

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Chapter 1

Solution of Partial Differential Equations

1.1 The General Solution of PDE

The general solution of a partial differential equation (PDE) is considered as a collection of all possible solutions of a given equation.

Example 1.1 *Consider the following first order linear PDE equations*

$$u_x(x, y) = 2\,x + y, \quad -\infty < x, y < \infty \tag{1.1}$$

$$u_y(x, y, z) = x + 2\,y + z, \quad -\infty < x, y, z < \infty \tag{1.2}$$

Solution. Let the variable y in (1.1) be fixed, and let us integrate both sides of equation (1.1) with respect to the variable x. Then, we obtain

$$u(x, y) = x^2 + y\,x + f(y) \tag{1.3}$$

for arbitrary differentiable function $f(y)$. Thus, all solutions of equation (1.1) are of the form (1.3), where $f(y)$ is any differentiable function.
Similarly, let us integrate both sides of the equation (1.2) with respect the variable y, when the variables x and z are fixed. Then, we obtain all solutions of equation (1.2) in following form:

$$u(x, y, z) = x + y^2 + z\,y + g(x, z), \tag{1.4}$$

for arbitrary differentiable function $g(x, z)$.

Example 1.2 *Consider the following second and third order linear PDE equations*

$$u_{xy}(x, y) = 2\,x + y, \quad -\infty < x, y < \infty \tag{1.5}$$

$$u_{xyz}(x, y, z) = x + 2\,y + z, \quad -\infty < x, y, z < \infty \tag{1.6}$$

Solution. Let us integrate both sides of equation (1.5) with respect to the variable y, when the variable x is fixed. Then, we obtain

$$u_x(x, y) = 2\ x\ y + \frac{1}{2}y^2 + f(x) \tag{1.7}$$

for arbitrary differentiable function $f(x)$. Next, we integrate equation (1.7) with respect to the variable x, when the variable y is fixed. Then, we obtain

$$u(x, y) = x^2 y + \frac{1}{2}y^2\ x + F(x) + g(y) \tag{1.8}$$

where $F(x)$ is an antiderivative to the function $f(x)$ and $g(y)$ is an arbitrary differentiable function of the variable y. Thus, all solutions of equation (1.5) are of the form (1.8), where $F(x)$ is an antiderivative to the arbitrary function $f(x)$, and $g(y)$ is any differentiable function of the variable y.
Now, let us integrate both sides of equation (1.6) with respect to the variable z, when the variables x and y are fixed. Then, we obtain

$$u_{xy}(x, y, z) = x\ z + 2y\ z + \frac{1}{2}z^2 + f(x, y) \tag{1.9}$$

Next, we integrate equation (1.9) with respect to the variable y, when the variables x and z are fixed. Then, we obtain

$$u_x(x, y, z) = x\ y\ z + y^2\ z + \frac{1}{2}z^2 y + F(x, y) + g(x, z) \tag{1.10}$$

where $F(x, y)$ is an antiderivative to $f(x, y)$ with respect to the variable y, and $g(x, z)$ is an arbitrary differentiable function of the variables x and y.
Finally, we integrate equation (1.10) with respect to the variable x, when the variables y and z are fixed. Then, we obtain

$$u(x, y, z) = \frac{1}{2}x^2\ y\ z + x\ y^2\ z + \frac{1}{2}x\ yz^2 + FF(x, y) + G(x, z) \tag{1.11}$$

where $FF(x, y)$ is an antiderivative to the antiderivative $F(x, y)$ with respect to the variable x, and $G(x, z)$ is an antiderivative to $g(x, z)$ with respect to the variable x.

Example 1.3 *Find all solutions of the following non-linear PDE equation:*

$$(u_{xx})^2 + (u_{yy})^2 = 0, \qquad -\infty < x, y < \infty \tag{1.12}$$

Solution. We note that $u_{xx} = 0$ and $u_{yy} = 0$. All solutions of the equation $u_{xx} = 0$ are in the form

$$u(x, y) = f(y)x + g(y) \tag{1.13}$$

and all solutions of the equation $u_{yy} = 0$ are in the form

$$u(x, y) = q(x)y + r(x), \tag{1.14}$$

for arbitrary differentiable functions $f(y)$, $g(y)$, $q(x)$ and $r(x)$. Then, all solutions of equation (1.12) have common part which is included in both (1.13) and (1.14). So that, the solutions which have both forms are in the following form:

$$u(x, y) = a\,x\,y + b\,x + c\,y + d,$$

for arbitrary constants a, b, c and d.

Example 1.4 *Consider the following first order PDE equation*

$$x\,u_x - 2\,x\,u_y = u, \qquad -\infty < x, y < \infty. \tag{1.15}$$

Show that

$$u(x, y) = xf(2\,x + y) \tag{1.16}$$

is the solution of equation (1.15) when f is any differentiable function. Find the solution within the family of solutions (1.16), which satisfies the condition

$$u(1, y) = y^2, \qquad -\infty < y < \infty.$$

Solution. Let us note that $f(2x + y)$ is the function of one variable $t = 2x + y$, $-\infty < t < \infty$. By differentiation, we find

$$u_x(x, y) = f(2x + y) + 2\,x\,f^{'}(2x + y), \qquad u_y(x, y) = x\,f^{'}(2x + y).$$

Hence, we compute

$$x\,u_x - 2\,x\,u_y = x\,f(2x+y) + 2\,x^2 f^{'}(2x+y) - 2\,x^2\,f^{'}(2x+y) = x\,f(2x+y) = u(x, y).$$

Now, applying the condition $u(1, y) = y^2$ to the solution (1.16), we find function f. Thus, $u(1, y) = 1$, $f(2 * 1 + y) = y^2$. Let $t = 2 + y$ and $y = t - 2$. Then, $f(t) = (t - 2)^2$. We can choose $f(2x + y) = (2x + y - 2)^2$. Let us note that the solution $u(x, y) = x(2x + y - 2)^2$ satisfies the condition $u(1, y) = y^2$.

1.2 First Order PDE with Constant Coefficients

Let us consider the following equation

$$a\,u_x + b\,u_y + c\,u = f(x, y), \qquad a^2 + b^2 > 0. \tag{1.17}$$

where a, b, and c are constant coefficients, and $f(x, y)$ is a given continuous function.
Let us consider the case when $b \neq 0$. Then, we shall transform the equation

$$a\,u_x + b\,u_y + c\,u = f(x, y),$$

given in x, y coordinates to the equation

$$b\,v_z + c\,v = f(\frac{w + az}{b}, z)$$

in the new coordinates $w,\ z$

$$w = b\,x - a\,y, \quad z = y$$

Hence, we find

$$x = \frac{w + a\,z}{b}, \qquad y = z.$$

In terms of the new coordinates, we compute

$$a\,u_x + b\,u_y = a(v_w w_x + v_z z_x) + b(v_w w_y + v_z z_y) = (a\,b - b\,a)v_w + b\,v_z = b\,v_z.$$

Thus, in the new variables, equation (1.17), takes the form

$$b\,v_z + c\,v = f(\frac{w + a\,z}{b}, z) \tag{1.18}$$

Now, we shall solve the equation

$$b\,v_z + c\,v = g(w, z), \tag{1.19}$$

for $g(w, z) = f(\dfrac{w + a\,z}{b}, z)$

In order to find the general solution of equation (1.19), we divide the above equation by b and multiplying by the factor $e^{\frac{cz}{b}}$, to obtain

$$e^{\frac{cz}{b}}\,v_z(w, z) + e^{\frac{cz}{b}}\frac{c}{b}v(w, z) = \frac{1}{b}g(w, z)e^{\frac{cz}{b}},$$

or

$$\frac{\partial}{\partial z}[e^{\frac{cz}{b}}\,v(w, z)] = \frac{1}{b}g(w, z)e^{\frac{cz}{b}}. \tag{1.20}$$

Integrating both sides of equation (1.20) with respect to z, and multiplying by the factor $e^{-\frac{cz}{b}}$, we obtain the following general solution of equation (1.19)

$$v(w, z) = e^{-\frac{cz}{b}}[\frac{1}{b}\int g(w, z)e^{\frac{cz}{b}}\,dz + C(w)], \tag{1.21}$$

where $C(w)$ is an arbitrary differentiable function of the variable z.

In the case when $b = 0$, we have already the equation in the form (1.19), so that

$$au_x + c\,u = f(x, y).$$

The new function

$$v(w,z) \equiv u(x,y) = u(\frac{w + a\,z}{b}, z).$$

Now, we can solve equation (1.19) by formula (1.21) to get the solution $v(w,z)$, and then to obtain the solution $u(x,y) = v(b\,x - a\,y, y)$. Below, we shall present some examples following the above solution of the first order linear equation with constant coefficients.

Example 1.5 .

(1a) Find all solutions of the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + u = 1 \qquad (1.22)$$

(1b) Find the solution of equation (1.22) which satisfies the condition

$$u(x,y) = 2 \quad for \quad y = x$$

Solution 1a. Note that the coefficients $a = 1,\ b = -1,\ c = 1$ and the function $f(x,y) = 1$. First, we transfer the equation to the form

$$v_z + c\,v = g(w,z),$$

by the mapping

$$w = -\,x -\,y, \qquad z = y$$

So that we have

$$x = -w - z, \qquad y = z.$$

We consider the new unknown

$$v(w,z) = u(x,y) = u(-w - z, z),$$

for which, we compute the expression

$$u_x - u_y = (v_w\,w_x + w_z\,z_x) - (v_w\,w_y + w_z\,z_y) = -v_z.$$

Since $g(w,z) = f(-w - z, z) = 1$, therefore, we obtain the equation

$$-v_z(w,z) + v(w,z) = 1.$$

Multiplying the above equation by e^{-z}, we have

$$-e^{-z}v_z(w,z) + e^{-z}v(w,z) = e^{-z} \qquad \text{or} \qquad -\frac{\partial}{\partial z}[e^{-z}\,v] = e^{-z}.$$

By integration with z

$$-e^{-z}v = -e^{-z} + C(w) \quad \text{or} \quad v(w,z) = 1 - C(w)e^{z}$$

Hence, we find the solution

$$v(w,z) = u(-w-z,z) = 1 - C(w)\, e^{z} \quad \text{and} \quad u(x,y) = 1 - C(-x-y)e^{y},$$

for arbitrary differentiable function $C(w)$.
Solution 1b. For $y = x$, we find $u(x,x) = 1 - C(-2x)e^{x} = 2$.
So that

$$C(-2x) = -e^{-x}$$

Let $t = -2x$. Then, we have $C(t) = -e^{-\frac{t}{2}}$ and the solution

$$u(x,y) = 1 + e^{\frac{-x-y}{2}} e^{y} = 1 + e^{\frac{y-x}{2}}$$

satisfies the condition $u(x,x) = 2$.

Example 1.6 *Find the general solution of the equation*

$$3\, u_x - 2\, u_y + u = x. \tag{1.23}$$

Solution. We consider the new variables

$$w = 2\,x + 3\,y, \qquad z = y.$$

Hence

$$x = \frac{w - 3\,z}{2}, \qquad y = z.$$

Then, we introduce the unknown

$$v(w,z) = u(x,y) = u(\frac{w-3z}{2}, z).$$

Now, we compute

$$3\,u_x - 2\,u_y = 3(v_w w_x + v_z z_x) - 2(v_w w_y + v_z z_y) = -2v_z.$$

Equation (1.23), in the new variables becomes

$$-2v_z + v = \frac{1}{2}(w - 3\,z). \tag{1.24}$$

Dividing by -2 and multiplying by the factor $e^{-\frac{z}{2}}$, we obtain

$$\frac{\partial}{\partial z}[e^{-\frac{z}{2}} v(w,z)] = -\frac{1}{4}e^{-\frac{z}{2}}(w - 3\,z). \tag{1.25}$$

Integrating both sides of (1.25) with respect z, when w is fixed, we find

$$\begin{aligned} e^{-\frac{z}{2}}v(w,z) &= -\frac{1}{4}w\int e^{-\frac{z}{2}}\,dz+\frac{3}{4}\int ze^{-\frac{z}{2}}\,dz+C(w)\\ &= \frac{1}{2}we^{-\frac{z}{2}}+\frac{3}{4}[ze^{-\frac{z}{2}}(-2)-\int e^{-\frac{z}{2}}(-2)\,dz]+C(w) \qquad (1.26)\\ &= e^{-\frac{z}{2}}[\frac{w}{2}-\frac{3z}{2}-3]+C(w), \end{aligned}$$

where $C(w)$ is an arbitrary differentiable function of the variable w.
Hence, we find the solution

$$v(w,z)=\frac{1}{2}[w-3\,z-6]+e^{\frac{z}{2}}C(w).$$

and coming back to the original variables, we obtain the general solution of equation (1.23) in the following form

$$u(x,y)=\frac{1}{2}[2x+3y-3y-6]+e^{\frac{y}{2}}C(2x+3y)=x-3+e^{\frac{y}{2}}C(2x+3y).$$

Let us observe that choosing the function $C(2x+3y)$, we obtain a particular solution. For example, the particular solution is

$$u(x,y)=x-3+e^{\frac{y}{2}}.$$

for $C(2x+3y)=1$, Indeed, we have

$$3u_x-2u_y+u=3+e^{\frac{y}{2}}+x-3+e^{\frac{y}{2}}=x.$$

Also, for $C(2x+3y)=2x+3y$, we have the particular solution

$$u(x,y)=x-3+e^{\frac{y}{2}}(2x+3y).$$

1.3 Exercises

Question 1. Find the general solution of the equations

(a) $u_x=3x+2y$,

(b) $u_{xy}=x\,y$,

(c) $u_{xyz}=x+y+z$.

Question 2. Find all solutions of the equations

(a) $u_x - 2u_y + u = x + y$,

(b) $u_x + 2u_y + 3u = x + y$,

(c) $u_x - u_y + u = 0$.

Question 3.

1. Find all solutions of the equation

$$u_x + u_y - 2u = y$$

2. Find the solution of the equation which satisfies the condition

$$u(x,1) = x \qquad \text{for} \qquad -\infty < x < \infty$$

Chapter 2

Classification of Partial Differential Equations of the Second Order

2.1 Hyperbolic, Elliptic and Parabolic Equations

We shall consider the following form of partial differential equations:

$$\begin{aligned} Lu \ &\equiv a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + \\ &+d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{py} + g(x,y)u = f(x,y), \end{aligned} \tag{2.1}$$

where $u(x,y)$ is an unknown function and the coefficients

$$a(x,y),\ b(x,y),\ c(x,y),\ d(x,y),\ e(x,y),\ g(x,y)$$

and the right side $f(x,y)$ are given functions of the variables (x,y) in the domain Ω.
For the classification purpose, we consider the following differential operator of the second order associated with the main part of equation (2.1)

$$L_0 u = a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}$$

We shall observe that the differential operator

$$L_1 u = d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + g(x,y)u.$$

of order one does not effect the type of the equation. The type of an equation is determined by the operator L_0 of the sedcond order.
Classification. All the equations of the general form (2.1) are divided in three the following classes pending on the sign of the discriminant $b^2 - a\ c$.

1. (2.1) is called hyperbolic equation if the discriminant $b^2 - a\ c > 0$, for all $(x, y) \in \Omega$,

2. (2.1) is called elliptic equation if the discriminant $b^2 - a\ c < 0$, for all $(x, y) \in \Omega$,

3. (2.1) is called parabolic equation if the discriminant $b^2 - a\ c = 0$, for all $(x, y) \in \Omega$.

Also,

1. the operator L is called

   hyperbolic operator if the discriminant $b^2 - a\ c > 0$,

2. the operator L is called elliptic operator if the discriminant $b^2 - a\ c < 0$,

3. the operator L is called parabolic operator if the discriminant $b^2 - a\ c = 0$.

Example 2.1 .

The wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

is the hyperbolic equation, since the discriminant

$$b^2 - a\ c = 0^2 - 1(-1) = 1 > 0.$$

- *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is the elliptic equation, since the discriminant

$$b^2 - a\ c = 0^2 - 1 * 1 = -1 < 0.$$

- *The heat equation*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

is the parabolic equation, since the discriminant

$$b^2 - a\ c = 0^2 - 1 * 0 = 0.$$

2.2 The Standard Form of Hyperbolic, Elliptic and Parabolic Equations

The following standard or canonical forms of hyperbolic, elliptic and parabolic equations are considered:

1. The first standard form of a hyperbolic equation
$$\frac{\partial^2 u}{\partial t \partial x} = f^*(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}),$$
2. the second standard form of a hyperbolic equation
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f^*(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}),$$
3. the standard form of an elliptic equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f^*(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}),$$
4. the standard form of a parabolic equation
$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} + f^*(t, x, u, \frac{\partial u}{\partial x}),$$

Here, f^* is a function independent of the second derivatives.
In order to transform equation (2.1) into its canonical form, we consider the new variables
$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y).$$
For the composed function $u(\xi, \eta) = u(\xi(x, y), \eta(x, y))$, we compute the following derivatives:
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$
and
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}(\frac{\partial \xi}{\partial x})^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial x})^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + $$
$$+ \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}(\frac{\partial \xi}{\partial y})^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial y})^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}$$

Now, substituting the above relationship to equation (2.1), we obtain the following equation in terms of the variables ξ and η:

$$A\frac{\partial^2 u}{\partial \xi^2} + 2B\frac{\partial^2 u}{\partial \xi \partial \eta} + C\frac{\partial^2 u}{\partial \eta^2} + D\frac{\partial u}{\partial \xi} + E\frac{\partial u}{\partial \eta} + Gu = F \tag{2.2}$$

where

$$\begin{aligned}
A &= a(\frac{\partial \xi}{\partial x})^2 + 2b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c(\frac{\partial \xi}{\partial y})^2\\
B &= a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}) + c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\\
C &= a(\frac{\partial \eta}{\partial x})^2 + 2b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c(\frac{\partial \eta}{\partial y})^2\\
D &= a\frac{\partial^2 \xi}{\partial x^2} + 2b\frac{\partial^2 \xi}{\partial x \partial y} + c\frac{\partial^2 \xi}{\partial y^2} + d\frac{\partial \xi}{\partial x} + e\frac{\partial \xi}{\partial y}\\
E &= a\frac{\partial^2 \eta}{\partial x^2} + 2b\frac{\partial^2 \eta}{\partial x \partial y} + c\frac{\partial^2 \eta}{\partial y^2} + d\frac{\partial \eta}{\partial x} + e\frac{\partial \eta}{\partial y}\\
G &= g, \qquad F = f.
\end{aligned}$$

It may be verified that

$$B^2 - A\,C = (b^2 - a\ c)(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x})^2,$$

where

$$J(\xi,\eta) = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \neq 0,$$

is Jacobian of the mapping

$$\xi = \varphi(x,y), \qquad \eta = \psi(x,y), \tag{2.3}$$

Thus, the type of the equation remains the same in the new coordinates ξ and η, provided that Jacobian $J(\xi,\eta) \neq 0$.
We note that, every equation of the general form (2.1) can be transformed by a transformation

$$\xi = \varphi(x,y), \qquad \eta = \psi(x,y). \tag{2.4}$$

to a canonical form.

2.3 Transformation of a Hyperbolic Equation into a Standard-Canonical Form

If we assume the canonical form of the hyperbolic equation in the two variables ξ, η

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = f^*(t, \xi, ete, u), \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f^*(t, \xi, \eta, u),$$

with the wave equation as the representative

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{\partial \eta^2}.$$

then, we have to put the following conditions:

$$\begin{aligned} A &= a(\frac{\partial\xi}{\partial x})^2 + 2b\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + c(\frac{\partial\xi}{\partial y})^2 = 0, \\ C &= a(\frac{\partial\eta}{\partial x})^2 + 2b\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + c(\frac{\partial\eta}{\partial y})^2 = 0, \end{aligned} \tag{2.5}$$

We shall call the curves given by equations (2.5) as characteristics of equation (2.1), if the functions given in the implicit form

$$\varphi(x, y) = constant \qquad \text{and} \qquad \psi(x, y) = constant$$

are different solutions of characteristic equations (2.5).
Then, along of the characteristic curves the following equations hold:

$$\begin{aligned} \frac{\partial\xi}{dx}dx + \frac{\partial\xi}{dy}dy = 0. \\ \frac{\partial\eta}{dx}dx + \frac{\partial\eta}{dy}dy = 0, \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\xi_x}{\xi_y}, \\ \frac{dy}{dx} &= -\frac{\eta_x}{\eta_y}, \end{aligned} \tag{2.7}$$

Substituting relation (2.7) between $\frac{dy}{dx}$ and $\xi_x, \eta_x, \xi_y, \eta_y$, to equation (2.5), we obtain the following ordinary differential equation

$$a(\frac{dy}{dx})^2 - 2\, b\frac{dy}{dx} + c = 0. \tag{2.8}$$

Hence, we obtain two ordinary differential equations to determine the characteristics curves:

$$\frac{dy}{dx} = \lambda_1 = \frac{b(x,y) - \sqrt{b(x,y)^2 - a(x,y)c(x,y)}}{a(x,y)}$$
$$\frac{dy}{dx} = \lambda_2 = \frac{b(x,y) + \sqrt{b(x,y)^2 - a(x,y)c(x,y)}}{a(x,y)} \tag{2.9}$$

where λ_1 and λ_2 are roots of the quadratic equation

$$a\lambda^2 - 2b\lambda + c = 0. \tag{2.10}$$

and

$$\frac{\xi_x}{\xi_y} = -\lambda_1,$$
$$\frac{\eta_x}{\eta_y} = -\lambda_2 \tag{2.11}$$

Example 2.2 .

(a) *Find equations of the characteristic for the following hyperbolic equation:*

$$y^2 u_{xx} - x^2 u_{yy} = 0, \qquad x > 0,\ y > 0. \tag{2.12}$$

(b) *Transform equation (2.12) into the canonical form*

Solution
To (a). The two characteristic equations are

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} = -\frac{x}{y}, \qquad \frac{dy}{dx} = \frac{b = \sqrt{b^2 - ac}}{a} = \frac{x}{y} \tag{2.13}$$

Note that this is equivalent to setting $A = C = 0$.
Solving the above ordinary differential equations by the method of separating variables, we find the equations for the characteristics in the implicit form

$$y^2 - x^2 = constant, \qquad y^2 + x^2 = constant.$$

To (b). We find the canonical form of the hyperbolic equation in the new variables

$$\xi = y^2 - x^2, \qquad \eta = y^2 + x^2.$$

Then, we compute the coefficients A, B, C, D, E, F, G in the equation

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu = G. \tag{2.14}$$

having the coefficients $a = y^2,\ b = 0,\ c = -x^2,\ d = 0,\ e = 0,\ f = 0,\ g = 0$ in the equations

$$au_{\xi\xi} + au_{\xi\eta} + au_{\eta\eta} + au_{\xi} + eu_{\eta} + fu = g.$$

we compute

$$A = a(\xi_x)^2 + 2\ b\ \xi_x\xi_y + c\ (\xi_y)^2 = y^2(-2x)^2 - x^2(2y)^2 = 0$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = y^2(-2x)(2x) - x^2(2y)(2y) = -8x^2y^2$$

$$C = a(\eta_x)^2 + 2\ b\ \eta_x\eta_y + c\ (\eta_y)^2 = y^2(-2x)^2 - x^2(2y)^2 = 0$$

$$D = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -2((x^2 + y^2)$$

$$E = a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 2(y^2 - x^2)$$

$$F = 0, \quad G = 0.$$

Substituting the above coefficients to the equation (2.14), we obtain the following equation

$$u_{\xi\eta} = \frac{-(x^2 + y^2)u_\xi + (y^2 - x^2)u_\eta}{4x^2y^2}. \tag{2.15}$$

Hence, by the equations of the characteristics, we find the first canonical form

$$u_{\xi\eta} = \frac{\eta u_\xi - \xi u_\eta}{2(\xi^2 - \eta^2)}$$

We can find the second canonical form of equation (2.14), introducing the new variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta.$$

Now, we can rewrite equation (2.15), given in variables ξ, η in terms of the variables α, β to obtain the second canonical form.
Then, we compute

$$u_\xi = u_\alpha\alpha_\xi + u_\beta\beta_\xi = u_\alpha + u_\beta$$

$$u_\eta = u_\alpha\alpha_\eta + u_\beta\beta_\eta = u_\alpha - u_\beta$$

$$u_{\xi\eta} = u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta + u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta = u_{\alpha\alpha} - u_{\beta\beta}$$

Hence, by substituting, we obtain the second canonical from equation (2.12).

$$u_{\alpha\alpha} - u_{\beta\beta} = -\frac{\beta u_\alpha + \alpha u_\beta}{2\alpha\beta}.$$

Example 2.3 . *Consider th equation*

$$y\frac{\partial^2 u}{\partial x^2} + (x + y)\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial^2 u}{\partial y^2} = 0, \tag{2.16}$$

1. *Find the range of x and y for which the equation is hyperbolic.*
2. *Transform the equation to a canonical form.*

Solution. We have $a = y,\ b = \dfrac{1}{2}(x+y)\ c = x,\ d = e = g = f = 0$. The discriminant

$$b^2 - a\,c = \frac{1}{4}(x-y)^2 > 0,$$

is positive for all real $x \neq y$.
Then, equation (2.16) is hyperbolic on the whole x, y plane, except the line $y = x$, where (2.16) becomes the parabolic equation.
The equation for the characteristics is:

$$y(\frac{dy}{dx})^2 - (x+y)\frac{dy}{dx} + x = 0. \tag{2.17}$$

Then, we find the roots

$$r_1 = \frac{(x+y) - \sqrt{(x+y)^2 - 4xy}}{2\,y} = \frac{(x+y) - |x-y|}{2\,y} = \begin{cases} \dfrac{x}{y}, & if \;\; x \leq y \\ 1, & if \;\; x > y, \end{cases}$$

$$r_2 = \frac{(x+y) - \sqrt{(x+y)^2 - 4xy}}{2\,y} = \frac{(x+y) + |x-y|}{2\,y} = \begin{cases} \dfrac{x}{y}, & if \;\; x \geq y \\ 1, & if \;\; x > y, \end{cases}$$

So that, we consider $\lambda_1 = \dfrac{x}{y}$ and $\lambda_2 = 1$.
Hence, we obtain the following two equations for characteristics

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \frac{dy}{dx} = 1 \tag{2.18}$$

Solving the above equations, we find

$$\varphi(x,y) = y^2 - x^2 = constant, \qquad \psi(x,y) = y - x = constant. \tag{2.19}$$

Now, we consider the mapping

$$\xi = y^2 - x^2, \qquad \eta = y - x. \tag{2.20}$$

To transform the hyperbolic equation to the standard form, we compute

$$A = a(\xi_x)^2 + 2b\xi_x\xi_y + c\,(\xi_y)^2 = y(-2x)^2 + (x+y)(-2x)(2y) + (2y)^2 = 0$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \imath_y\eta_x) + c\xi_y\eta_y =$$

$$= y(-2x)(-1) + \tfrac{1}{2}(x+y)(-2x-2y) + 2xy = -(x-y)^2 = -\psi^2,$$

$$C = a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 = y - (x+y) + x = 0$$

$$D = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -2(y-x) = -2\psi,$$

$$E = a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0, \qquad F = 0.$$

In the new variables ξ and η, the hyperbolic equation (2.16) takes the standard form

$$\eta \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u}{\partial \xi} = 0. \tag{2.21}$$

2.4 Transformation of an Elliptic Equation into the Standard-Canonical Form

In the case of elliptic equations, when the discriminant $b^2 - ac < 0$, the roots λ_1 and λ_2 of the quadratic equation (2.10) are complex and hence $\varphi(x, y)$ and $\psi(x, y)$ will be also complex functions. Therefore, mapping (2.3) is determined by the conjugate roots of quadratic equation (2.10), that is, by $\lambda - i\mu$ and $\lambda + i\mu$. Then , the functions $\varphi(x, y)$ and $\psi(x, y)$ are solutions of the differential equations

$$\frac{dy}{dx} = \lambda(x, y) - i\mu(x, y), \qquad \frac{dy}{dx} = \lambda(x, y) + i\mu(x, y) \tag{2.22}$$

Thus, if $\varphi(x, y) = \alpha(x, y) + i\beta(x, y) = \mathit{constant}$ is the solution of the characteristic equation then the conjugate $\psi(x, y) = \alpha(x, y) - i\beta(x, y)$ is also the solution of the characteristic equation. We consider the mapping

$$\varphi(x, y) = \alpha(x, y) + i\beta(x, y), \qquad \psi(x, y) = \alpha - i\beta(x, y).$$

Choosing real and imaginary parts as the new variables $\xi = \alpha(x, y)$, $\eta = \beta(x, y)$, by the formulae (2.2), we find that $A = C$ and $B = 0$. So that, the canonical form of an elliptic equation is

$$A(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}) + D\frac{\partial u}{\partial \xi} + E\frac{\partial u}{\partial \eta} + Gu = F \tag{2.23}$$

or dividing by A, we obtain the canonical form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{D}{A}\frac{\partial u}{\partial \xi} + \frac{E}{A}\frac{\partial u}{\partial \eta} + \frac{G}{A}u = \frac{F}{A}, \tag{2.24}$$

with the Laplace's equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0,$$

as the representative.

Example 2.4 . *Determine type of the equation*

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0, \qquad y > 0, \tag{2.25}$$

Transform the equation into the canonical form.

Solution. The discriminant $b^2 - ac = -y < 0$ for $y > 0$. So that, it is the elliptic equation. The characteristics of this equation are defined by the equations

$$(\frac{dy}{dx})^2 + y = 0, \qquad \frac{dy}{dx} = i\sqrt{y}, \qquad \frac{dy}{dx} = -i\sqrt{y},$$

Solving the above equations, we obtain the following mapping

$$\varphi(x,y) = x + 2i\sqrt{y}, \qquad \psi(x,y) = x - 2i\sqrt{y}.$$

Let $\xi = x$ and $\eta = 2\sqrt{y}$. Then, we find

$$\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} \\ &\frac{\partial u}{\partial y} = \frac{1}{\sqrt{y}}\frac{\partial u}{\partial \eta}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{1}{y}\frac{\partial^2 u}{\partial \eta^2} - \frac{1}{2\sqrt{y^3}}\frac{\partial u}{\partial \eta} \end{aligned} \tag{2.26}$$

Hence, we find in terms of ξ, η

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0, \qquad y > 0.$$

and the standard form of the equation in the new variables is:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0, \qquad y > 0.$$

2.5 Transformation of a Parabolic Equation into the Standard-Canonical Form

For a parabolic equation there is only one repeating root of equation (2.10) equal to $\frac{b}{a}$. Then, we find only one solution $\varphi(x,y) = constant$ of the equation

$$\frac{dy}{dx} = \frac{b}{a}$$

In this case, we consider the mapping

$$\xi = \varphi(x,y), \qquad \eta = \psi(x,y),$$

where $\psi(x,y) = constant$ is an arbitrary family of curves such that Jacobian $J(\varphi,\psi) = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$.
Because

$$\frac{\varphi_x}{\varphi_y} = -\lambda = -\frac{b}{a}$$

therefore

$$a\varphi_x + b\varphi_y = 0.$$

Multiplying this equation by b and using the equality $b^2 = ac$, we find

$$ab\varphi_x + b^2\varphi_y = 0$$

$$ab\varphi_x + ac^2\varphi_y = 0$$

$$a(b\varphi_x + c\varphi_y) = 0 \qquad b\varphi_x + c\varphi_y = 0$$

Hence, we have

$$B = a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\varphi_y = \psi_x(a\varphi_x + b\varphi_y) + \psi_y(b\varphi_x + c\varphi_y) = 0.$$

Therefore, the term with $\dfrac{\partial^2 u}{\partial\xi\partial\eta}$ in the canonical form is absent.

Example 2.5 *Transform to the standard form the following equation:*

$$u_{tt} - 2u_{tx} + u_{xx} - u_t - u_x + u = 0, \qquad t \geq 0, \quad -\infty \leq x \leq \infty.$$

Solution. From the general form of a linear PDE of the second order (2.1), we find the coefficients

$$a = 1,\ b = -1,\ c = 1,\ d = -1,\ e = -1,\ g = 1,\ f = 0.$$

In order to determine the type of the equation, we compute the discriminant

$$b^2 - a\,c = (-1)^2 - 1 = 0.$$

Since the discriminant equals to zero, the equation is parabolic one. Then, there is one family of characteristics determined by the ordinary differential equation

$$\frac{dx}{dt} = \frac{b}{a} = -1$$

Hence, we obtain the solution $t + x = constant$. Now, we choose the mapping

$$\xi = t + x, \qquad \eta = t.$$

Let us note that for η, we are free to choose any function for which the Jacobian

$$\xi_t\ \eta_x - \xi_x\eta_t \neq 0.$$

Now, we compute the coefficients

$$A = a(\xi_t)^2 + 2b\xi_t\ \xi_x + c(\xi_x)^2 = 1 - 2 + 1 = 0,$$

$$B = a\xi_t\ \eta_t + b(\xi_t\ \eta_x + \xi_x\ \eta_t) + c\ \xi_x\ \eta_x = 1 - (0 + 1) + 0 = 0$$

$$C = a\ (\eta_t)^2 + 2b\ \eta_t\eta_x + c(\eta_x)^2 = 1 - 2 * 0 + 0 = 1,$$

$$D = a\ \xi_{tt} + 2b\ \xi_{tx} + c\ (\xi_x)^2 + d\ \xi_t + e\ \xi_x = 0 - 2 * 0 + 0 - 1 - 1 = -2,$$

$$E = a\ \eta_{tt} + 2b\ \eta_{tx} + c\ (\eta_x)^2 + d\ \eta_t + e\ \eta_x = 0 - 2 * 0 + 0 - 1 - 0 = -1,$$

$$G = 1, \qquad F = 0.$$

Hence, by the general form (2.2), we obtain the standard form of the equation

$$u_{\eta\eta} - 2u_{\xi} - u_{\eta} + u = 0.$$

or

$$u_{\xi} = \frac{1}{2}u_{\eta\eta} + \frac{1}{2}u_{\eta} - \frac{1}{2}u.$$

Example 2.6 . *Determine type of the equation*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{2.27}$$

Transform the equation into the canonical form.

Solution. The discriminant $b^2 - ac = x^2y^2 - x^2y^2 = 0$, so that it is a parabolic equation for all real x and y.
In order to find the canonical form of the equation, we solve the characteristics equation

$$\frac{dy}{dx} = \frac{y}{x}.$$

It is easy to find the solution $y = kx$, where k is a generic constant. Then, we consider the mapping

$$\xi = y - kx, \qquad \eta = \psi(x, y),$$

Here, $\psi(x, y)$ is an arbitrary function such that Jacobian $\xi_x\psi_y - \xi_y\psi_x \neq 0$. Let $\psi(x, y) = x$. Then, we find the coefficients

$$A = x^2\xi_x^2 + 2xy\xi_x\xi_y + y^2\xi_y^2 = (kx - y)^2 = 0,$$

$$B = x^2\xi_x\eta_x + xy(\xi_x\eta_y + \xi_y\eta_x) + y^2\xi_y\eta_x = 0,$$

$$C = x^2\eta_x^2 + 2xy\eta_x\eta_y + y^2\eta_y^2 = x^2,$$

Hence, in the new variables ξ and η equation (2.27) takes the canonical form

$$\frac{\partial^2 u}{\partial \eta^2} = 0.$$

Applications of the canonical form of elliptic, parabolic and hyperbolic equations

1. The three major classifications as elliptic, parabolic and hyperbolic equations, in fact classify physical problems into three basic physical types: steady-state problems, diffusion and wave propagation. The mathematical solutions of these three types of equations are very different.

2. Much of the theoretical work on the properties of solutions to hyperbolic problems assume the equation has been written in the canonical form

$$u_{\xi\xi} - u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta).$$

3. Many computer programs have been written to find the numerical solution of the canonical form. Having the numerical solution in the new variables, we can always come back to the original variables.

2.6 Exercises

Question 2.1 *Classify the following equations:*

1.
$$9u_{xx} + 12u_{xy} + 4u_{yy} + u_x = 0.$$

2.
$$u_{xx} - 8u_{xy} + 2u_{yy} + xu_x - yu_y = 0,$$

3.
$$4u_{xx} + 2u_{xy} + u_{yy} - u_y = 0.$$

4. *Find characteristics curves of the above three equations.*

5. *Transform the above three equations into their canonical forms.*

Question 2.2 *Transform the equation*

$$u_{tx} + u_t + u_x + u = f(t,x),$$

into the equation

$$u_{\xi,\eta} - u_{\xi\xi} + u_\xi + u_\eta + u = g(\xi,\eta).$$

Question 2.3 *Find characteristics of Tricomi's equation*

$$y\ u_{xx} + u_{yy} = 0,$$

in the lower half-plane $y < 0$. Transform Tricomi's equation into the canonical form in the upper-half of the plane when $y > 0$.

Question 2.4 *Show that all linear partial differential equations of the second order in two variables x and y of elliptic type with constant coefficients can be transformed into the canonical form*

$$u_{xx} + u_{yy} + gu = f(x,y)$$

Question 2.5 *Show that all linear partial differential equations of the second order in two variables t and x of hyperbolic type with constant coefficients can be transformed into the canonical form*

$$u_{tt} - u_{xx} + gu = f(t,x)$$

Chapter 3

Hyperbolic Equations

3.1 The Initial Value Problem for Wave Equation

Find the solution $u(t, x)$ of the initial value problem

$$\begin{aligned} &u_{tt} = k^2 u_{xx}, \qquad -\infty < x < \infty, \\ &u(0, x) = \phi_0(x), \qquad u_t(0, x) = \phi_1(x), \qquad 0 < t < \infty. \end{aligned} \tag{3.1}$$

This problem, which has no boundaries, describes the motion of an infinite string with given initial conditions and was solved by French mathematician D'Alembert. The solution $u(t, x)$ is given by the D'Alembert formula

$$u(t, x) = \frac{1}{2}[\phi_0(x - kt) + \phi_0(x + kt)] + \frac{1}{2k}\int_{x-kt}^{x+kt} \phi_1(\xi)d\xi. \tag{3.2}$$

3.1.1 D'Alembert Solution

We shall solve initial value problem (3.1) into four steps.
Step1. We note that the characteristics equations for the wave equation (3.1) are

$$(\frac{dx}{dt})^2 - k^2 = 0, \qquad \frac{dx}{dt} = k, \qquad \frac{dx}{dt} = -k.$$

which have the solutions

$$x - kt = constant \qquad \text{and} \qquad x + kt = constant.$$

Let us write equation (3.1) in terms of the new variables

$$\xi = x - kt, \qquad \eta = x + kt.$$

to obtain the first canonical form of equation (3.1).

$$u_{\xi\eta} = 0. \tag{3.3}$$

Simple application of the chain rule gives

$$u_x = u_\xi + u_\eta$$
$$u_t = k(-u_\xi + u_\eta)$$
$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_{tt} = k^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

and, by substitution, it leads us to the first canonical form of equation (3.1)

$$u_{\xi\eta} = 0.$$

This completes step 1.
Step 2. In step 2, we integrate equation (3.3), firstly with respect to the variable ξ, to obtain the solution

$$u_\eta(\xi, \eta) = \phi(\eta)$$

and secondly, we integrate the equation with respect to the variable η to obtain

$$u(\xi, \eta) = \varphi(\eta) + \psi(\xi), \qquad \varphi(\eta) = \int \phi(\eta \, d\eta,$$

Thus, all solutions of equation (3.3) have the following form:

$$u(\xi, \eta) = \varphi(\eta) + \psi(\xi),$$

where $\phi(\eta)$ and $\psi(\xi)$ are differentiable arbitrary functions of the variables ξ and η.
For example, one can easily check that the functions

$$u(\xi, \eta) = \sin \eta + \xi^2$$
$$u(\xi, \eta) = \eta^2 + e^\xi$$
$$u(\xi, \eta) = \frac{1}{\eta} + \tan \xi,$$

are all solutions of $u_{\xi,\eta} = 0$. This completes step 2.
Step 3. In the step 3, we transform the solution $u(\xi, \eta)$ given in terms of the variables ξ and η to the original variables t and x
To find all solutions in terms of the original variables t and x, we substitute

$$\xi = x - kt, \qquad \eta = x + kt,$$

into

$$u(\xi, \eta) = \varphi(\eta) + \psi(\xi),$$

to obtain

$$u(t,x) = \varphi(x+kt) + \psi(x-kt). \tag{3.4}$$

The general solution (3.4) (all solutions) represents the sum of any two moving waves, each wave moves in opposite directions with velocity k. For example, the functions

$$u(t,x) = \sin(x-kt), \qquad one - right\ moving\ wave$$

$$u(t,x) = (x+kt)^2, \qquad one - left\ moving\ wave$$

$$u(t,x) = \sin(x-kt) + (x+kt)^2, \quad two\ oppositely\ moving\ waves$$

Step 4. In the step 4, we shall choose from all solutions that one which satisfies the initial-value conditions.
Thus, among all solutions of the form

$$u(t,x) = \varphi(x+kt) + \psi(x-kt) \tag{3.5}$$

with arbitrary differentiable functions φ and ψ, we choose that one which satisfies the initial-value conditions

$$u(0,x) = \phi_0(x), \qquad u_t(0,x) = \phi_1(x)$$

In order to find functions φ and ψ, we apply the initial conditions

$$\varphi(x) + \psi(x) = \phi_0(x), \qquad k\varphi^{'}(x) - k\psi^{'}(x) = \phi_1(x). \tag{3.6}$$

We now integrate the second equation of (3.6) to obtain a new expression in $\phi(x)$ and $\psi(x)$. Then, we solve algebraically the two equations. Then, by carrying out the integration on the second equation of (3.6) by integrating from x_0 to x, we obtain

$$\varphi(x) - \psi(x) = \frac{1}{k}\int_0^x \phi_1(\xi)\, d\xi + K. \tag{3.7}$$

From (3.6) and (3.7), we find

$$\begin{aligned} \varphi(x) &= \frac{1}{2}\phi_0(x) + \frac{1}{2k}\int_0^x \phi_1(\xi)\, d\xi + \frac{K}{2} \\ \psi(x) &= \frac{1}{2}\phi_0(x) - \frac{1}{2k}\int_0^x \phi_1(\xi)\, d\xi - \frac{K}{2}. \end{aligned} \tag{3.8}$$

Now, we substitute to formula (3.5),

$$\begin{aligned} \varphi(x+kt) &= \frac{1}{2}\phi_0(x+kt) + \frac{1}{2k}\int_0^{x+kt} \phi_1(s)\, ds + \frac{K}{2} \\ \psi(x-kt) &= \frac{1}{2}\phi_0(x-kt) - \frac{1}{2k}\int_0^{x-kt} \phi_1(s)\, ds - \frac{K}{2} \end{aligned}$$

Hence, the solution of the initial-value problem is

$$u(t,x) = \frac{1}{2}[\phi_0(x-kt) + \phi_0(x+kt)] + \frac{1}{2k}\int_{x-kt}^{x+kt} \phi_1(\xi)d\xi. \tag{3.9}$$

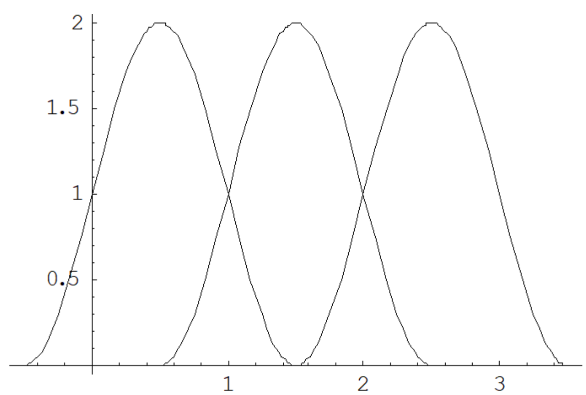
Examples

1. Let us consider the initial value problem

$$u_{tt} - k^2 u_{xx} = 0, \qquad t > 0, \quad -\infty < x < \infty,$$
$$u(0,x) = \sin x, \qquad u_t(0,x) = 0.$$

The sine wave would have the solution

$$u(t,x) = \frac{1}{2}[\sin(x-kt) + \sin(x+kt)]$$



2. Let us consider the initial problem with the initial velocity

$$u_{tt} - k^2 u_{xx} = 0, \qquad t > 0, \quad -\infty < x < \infty,$$
$$u(0,x) = 0, \qquad u_t(0,x) = \sin x.$$

The solution $u(t,x)$ is given below

$$u(t,x) = \frac{1}{2k}\int_{x-kt}^{x+kt} \sin\xi \, d\xi = \frac{1}{2k}[\cos(x+kt) - \cos(x-kt)].$$

3.1.2 The Initial Boundary Value Problem for Wave Equation

Let us consider the following initial boundary value problem:

$$\begin{aligned} &u_{tt} = k^2 u_{xx}, \qquad 0 < x < L, \qquad 0 < t < \infty, \\ &u(0,x) = \phi_0(x), \quad u_t(0,x) = \phi_1(x). \end{aligned} \tag{3.10}$$

When variable $x \in [0, L]$, the following three kind of boundary conditions are considered:

1. Controlled end points boundary conditions

$$u(t,0) = \psi_0(t),$$
$$u(t,L) = \psi_1(t).$$

2. Force specified on the boundaries

$$u_x(t,0) = \psi_0(t),$$
$$u_x(t,L) = \psi_1(t).$$

3. Elastic attachment

$$u_x(t,0) - \gamma_0 u(t,0) = \psi_0(t),$$
$$u_x(t,L - \gamma_1 u(t,L) = \psi_1(t).$$

3.2 Solution to the Finite Vibrating String by Separation of Variables

To solve the initial boundary value problem

$$\begin{aligned} &u_{tt} = k^2 u_{xx}, && 0 < x < L, && 0 < t < \infty, \\ &u(0,x) = \phi_0(x), && u_t(0,x) = \phi_1(x), && \\ &u(t,0) = 0, && u(t,L) = 0, && t \geq 0. \end{aligned} \tag{3.11}$$

we start by seeking standing wave solutions to the wave equation, that is, solutions of the following form:

$$u(t,x) = X(x)T(t)$$

Substituting this expression into the wave equation and separating variables gives us two ordinary differential equations

$$T^{''}(t) - k^2\lambda T(t) = 0, \qquad X^{''}(x) - \lambda X(x) = 0. \tag{3.12}$$

where, now the constant λ can be any real number.
In order to solve these ordinary differential equations, we find roots of the polynomials

$$P_2(\alpha) = \alpha^2 - k^2\lambda = 0, \qquad Q(\alpha) = \alpha^2 - \lambda = 0.$$

Then, we consider the following three cases:

Case 1. $\lambda < 0$.

If $\lambda < 0$ then there are two complex roots $\alpha_1 = -ik\sqrt{-\lambda}$ and $\alpha_2 = ik\sqrt{-\lambda}$. So that, for $\lambda = -\beta^2$, the solutions are

$$\begin{aligned} T(t) &= A\sin(k\beta t) + B\cos(k\beta t), \\ X(x) &= C\sin(\beta x) + D\cos(\beta x). \end{aligned} \tag{3.13}$$

where A, B, C and D are constants to be determined by the initial and boundary conditions.

Case 2. $\lambda = 0$

If $\lambda = 0$ then there are linear solutions to the equations (3.12)

$$T(t) = At + B, \qquad X(x) = Cx + D.$$

In this case the solution can be trivial ($u(t,x) \equiv 0$) or unbounded and feasible because of the initial value conditions.

Case 3. $\lambda > 0$.

If $\lambda = \beta^2 > 0$ then the solutions of equations (3.12) take the form

$$T(t) = Ae^{k\beta t} + Be^{-k\beta t}, \qquad X(x) = Ce^{\beta x} + De^{-\beta x}.$$

So, in this case, the solution either it is trivial ($u(t,x) = 0$) or unbounded because of initial boundary conditions.

Let us consider the solution given by formula (3.13), when $\lambda < 0$. Now, we apply the homogeneous boundary conditions plugging into $u(t,0) = u(t,L) = 0,\ t \geq 0$. Then, we obtain

$$u(t,0) = T(t)X(0) = D[A\sin(k\beta t) + B\cos(k\beta t)] = 0, \qquad D = 0,$$

$$u(t,L) = T(t)X(L) = C\sin(\beta L)[A\sin(k\beta t) + B\cos(k\beta t)] = 0, \quad \sin(\beta L) = 0.$$

The constant β has to satisfy the equation $\sin(\beta L) = 0$. So that, we find

$$\beta_n = \frac{n\pi}{L}, \qquad n = 1, 2, ...$$

We note that for

$$\begin{aligned} T_n(t) &= A_n\sin(k\beta_n t) + B_n\cos(k\beta_n t), \\ X_n(x) &= \sin(\beta_n x) \\ u_n(t,x) &= \sin(\beta_n x)[A_n\sin(k\beta_n t) + B_n\cos(k\beta_n t))] \end{aligned} \tag{3.14}$$

$u_n(t,x)$ is the solution of the wave equation which satisfies the homogeneous boundary conditions for arbitrary constants $A_n, B_n,\ \ n = 1, 2, ...;$

Because the wave equation is linear one, therefore every linear combination of $u_1(t,x), u_2(t,x), ...;$ is also a solution of the wave equation which satisfies the homogeneous boundary conditions. So that, the function

$$u(t,x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}[A_n \sin\frac{n\pi kt}{L} + B_n \cos\frac{n\pi kt}{L}] \tag{3.15}$$

satisfies the wave equation and homogenous boundary conditions.
Substituting sum (3.15) into the initial conditions

$$u(0,x) = \phi_0(x), \qquad u_t(0,x) = \phi_1(x),$$

gives the two equations

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = \phi_0(x), \qquad \sum_{n=1}^{\infty} A_n \frac{n\pi k}{L} \sin\frac{n\pi x}{L} = \phi_1(x) \tag{3.16}$$

Using the orthogonality condition

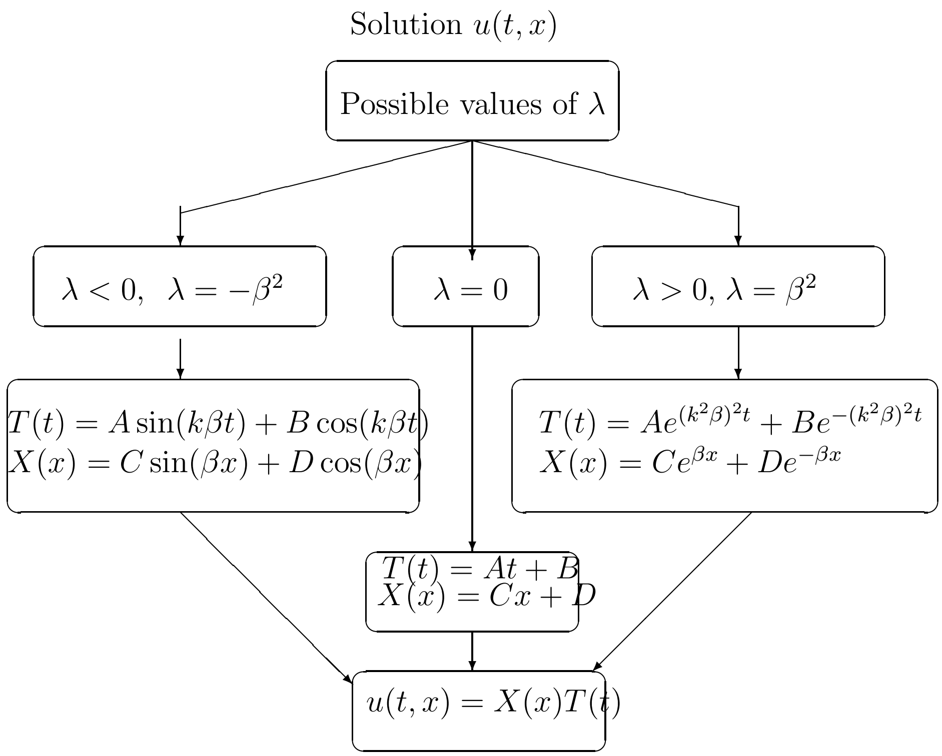
$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n \end{cases}$$

we can find the coefficients

$$\begin{aligned} A_n &= \frac{2}{n\pi k}\int_0^L \phi_1(x)\sin\frac{n\pi x}{L}dx, \\ B_n &= \frac{2}{L}\int_0^L \phi_0(x)\sin\frac{n\pi x}{L}dx. \end{aligned} \tag{3.17}$$

for $n = 1, 2, ...;$
Finally, the solution $u(t,x)$ of the initial boundary problem is given by formula (3.15) with the constants A_n and B_n, $n = 1, 2, ...;$ determined by the formulae (3.17).

We present these three cases below on the following diagram :



We shall now make the following observations:

1. Let us note that the solution takes the following form:

$$u(t,x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi kt}{L} \tag{3.18}$$

if the initial velocity $\phi_1(x) = 0$ with the homogeneous boundary conditions, when the initial position of the vibrating string $u(0,x) = \phi_0(x)$ is present.
Let the function

$$\phi_0(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Then, simple sine vibration of a string is given by the term

$$B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi kt}{L}.$$

Thus, adding the terms of each simple vibration, we obtain the solution of the initial boundary problem. Namely, consider the initial state of a vibrating string with fixed end points

$$\phi_0(x) = \sum_{n=1}^{m} b_n sin \frac{n\pi x}{L},$$

and with zero initial velocity $\phi_1(x) = 0$.
Then, the solution of such initial boundary value problem is

$$u(t,x) = \sum_{n=1}^{m} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi kt}{L}. \tag{3.19}$$

We can obtain formula (3.19) from the solution given by (3.18). Indeed, from the initial conditions, the coefficients are b_n for $n = 1, 2, ..., m$; and $b_n = 0$ for $n > m$. So, we compute the coefficients

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L \phi_0(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\sum_{s=1}^{m} b_s \int_0^L \sin\frac{s\pi x}{L}\sin\frac{n\pi x}{L}dx \\ &= b_n. \end{aligned}$$

Hence, by formula (3.18), we get the solution (3.19). For example, suppose that the initial string position is

$$\phi_0(x) = \sin\frac{\pi x}{L} + 0.5\sin\frac{3\pi x}{L} + 0.25\sin\frac{5\pi x}{L}.$$

The overall response to this initial condition would then be the sum of the responses to each term, that is

$$u(t,x) = \sin\frac{\pi x}{L}\cos\frac{\pi kt}{L} + 0.5\sin\frac{3\pi x}{L}\cos\frac{3\pi kt}{L} + 0.25\sin\frac{5\pi x}{L}\cos\frac{5\pi kt}{L}.$$

2. The n-th term in the solution (3.15)

$$\sin\frac{n\pi x}{L}[A_n \sin\frac{n\pi kt}{L} + B_n \cos\frac{n\pi kt}{L}]$$

is called n-th mode of vibration or n-th harmonic. This harmonic can be rewritten in the following form

$$R_n \sin\frac{n\pi x}{L}\cos\frac{n\pi k}{L}(t-\delta_n), \tag{3.20}$$

where δ_n is the phase angle and R_n is the amplitude. Indeed, we have

$$
\begin{aligned}
&[A_n \sin\frac{n\pi kt}{L} + B_n \cos\frac{n\pi kt}{L}] = \\
&\sqrt{A_n^2+B_n^2}[\frac{A_n}{\sqrt{A_n^2+B_n^2}} \sin\frac{n\pi kt}{L} + \frac{B_n}{\sqrt{A_n^2+B_n^2}} \cos\frac{n\pi kt}{L}] = \\
&\sqrt{A_n^2+B_n^2}[\sin\frac{n\pi k\delta_n}{L} \sin\frac{n\pi kt}{L} + \cos\frac{n\pi k\delta_n}{L} \cos\frac{n\pi kt}{L}] = \\
&\sqrt{A_n^2+B_n^2} \cos\frac{n\pi k}{L}(t-\delta_n) = R_n \cos\ \omega_n(t-\delta_n),
\end{aligned}
$$

where $\omega = \dfrac{n\pi k}{L}$ is frequency, $R_n = \sqrt{A_n^2+B_n^2}$ is the amplitude and δ_n is the phase angle.

Example 1.

1. By using separation variables and Fourier cosine series, solve the following problem for finite string with fixed ends for appropriate initial data $\phi_0(x)$ and $\phi_1(x)$

$$
\begin{aligned}
&u_{tt} = 4u_{xx}, \qquad 0 \le x \le L, \quad 0 < t < \infty, \\
&u(t,0) = 0, \qquad u(t,L) = 0, \\
&u(0,x) = \sum_{n=1}^{m} \frac{1}{2^n} \sin\frac{n\pi x}{L}, \qquad u_t(0,x) = 0.
\end{aligned}
\tag{3.21}
$$

2. Determine the frequency ω, the amplitude R_n and the phase angle δ_n .

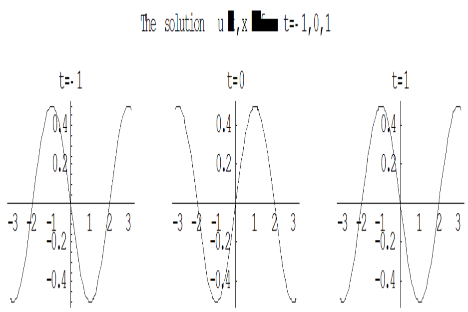
3. Graph the solution $u(t,x)$ for $m = 1,\ L = 2$ and $t = -1, 0, 1$

Solution.
To (a): By the formula (3.19), we find the solution

$$u(t,x) = \sum_{n=1}^{m} \frac{1}{2^n} \sin\frac{n\pi x}{L} \cos\frac{2n\pi t}{L}.$$

To (b). From the above formula, we find that the frequency is $\omega = \dfrac{2n\pi}{L}$, the amplitude is $R_n = \dfrac{1}{2^n}$ and the phase angle $\delta_n = 0$.
To (c): For $m = 1,\ L = 2$, the solution is

$$u(t,x) = \frac{1}{2} \sin\frac{\pi x}{2} \cos \pi t.$$



.

Example 2.
Solve the following initial boundary problem by the method of separation of variables:

$$
\begin{aligned}
&u_{tt} = 36u_{xx}, \quad t > 0, && 0 \le x \le 1, \\
&u(0,x) = 0, \quad u_t(0,x) = 4, && 0 \le x \le 1, \\
&u(t,0) = 0, \quad u(t,1) = 0, && t \ge 0.
\end{aligned}
\tag{3.22}
$$

Solution. We note that $\phi_0(x) = 0$. Therefore, by formula (3.17), the coefficint $B_n = 0$ and we compute the coefficients

$$
A_n = \frac{1}{3n\pi}\int_0^1 4\sin n\pi x\, dx = \frac{4}{3n^2\pi^2}[1-(-1)^n].
$$

Hence, by formula (3.15), the solution is:

$$
u(t,x) = \sum_{n=1}^{\infty} \frac{4}{3n^2\pi^2}[1-(-1)^n]\sin(n\pi x)\sin(6n\pi t).
$$

Example 3. What is the solution to the simple supported at the ends beam with initial conditions

$$
u(0,x) = \sin \pi x, \qquad u_t(0,x) = \sin \pi x, , \qquad 0 \le x \le 1.
$$

Solution. We note that the solution $u(t,x)$ satisfies the wave equation

$$
u_{tt} = k^2 u_{xx}, \qquad 0 \le x \le 1,
$$

with the homogeneous boundary conditions

$$
u(t,0) = 0, \qquad u(t,1) = 0, \qquad t \ge 0.
$$

and the initial value conditions

$$u(0,x) = \sin \pi x, \qquad u_t(0,x) = \sin \pi x, \qquad 0 \leq x \leq 1.$$

By method of separation variables, the solution is given by the formula

$$u(t,x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L}[A_n \sin \frac{n\pi kt}{L} + B_n \cos \frac{n\pi kt}{L}],$$

Hence, for $\phi_0(x) = \sin \pi x$, $\psi_1(x) = \sin \pi x$, and $L = 1$, we compute the coefficients using formulae (3.17)

$$A_n = \frac{2}{n\pi k}\int_0^1 \sin \pi x \sin n\pi x dx = \begin{cases} \frac{1}{k\pi}, & n = 1, \\ 0 & n \neq 1, \end{cases}$$

$$B_n = 2\int_0^1 \sin \pi x \sin n\pi x dx = \begin{cases} 1, & n = 1, \\ 0 & n \neq 1, \end{cases}$$

Thus, the solution $u(t,x)$ of the supported beam problem is

$$u(t,x) = sin\pi x[\frac{1}{\pi k}\sin(\pi kt) + \cos(\pi kt)],$$

for $0 \leq x \leq 1$ and $t \geq 0$.

Example 4. A gitar string of length $L = 1$ is pulled upward at middle so the it reaches heigt 0.5 and satisfies the wave equation

$$u_{tt} = 9u_{xx}, \qquad 0 \leq x \leq 1.$$

Assuming the initial position of the string

$$u(0,x) = \begin{cases} x, & 0 \leq x \leq 0.5, \\ 1-x, & 0.5 \leq x \leq 1, \end{cases}$$

and the initial speed of the string

$$u_t(0,x) = 1, \qquad 0 \leq x \leq 1.$$

Find the position $u(t,x)$ of the string at time t and point x.

Solution. We note that the solution $u(t,x)$ satisfies the wave equation

$$u_{tt} = 9u_{xx}, \qquad 0 \leq x \leq 1,$$

with the homogeneous boundary conditions

$$u(t,0) = 0, \qquad u(t,1) = 0, \qquad t \geq 0.$$

and the initial value functions

$$u(0,x) = \phi_0(x) = \begin{cases} x, & 0 \le x \le 0.5, \\ 1-x, & 0.5 \le x \le 1, \end{cases} \qquad u_t(0, x_= \phi_1(x) = 1. \qquad (3.23)$$

By method of separation variables the solution is given by the formula

$$u(t,x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}[A_n \sin\frac{n\pi kt}{L} + B_n \cos\frac{n\pi kt}{L},$$

Hence, for the given functions $\phi_0(x),\ \psi_1(x)$ by (3.23) and for $L = 1$, we (icients using formulae (3.17)

$$\begin{aligned} A_n &= \frac{2}{n\pi k}\int_0^1 \sin n\pi x dx = \frac{2}{kn\pi}(\frac{1}{n\pi} - \frac{\cos n\pi}{n\pi}) = \frac{2}{kn^2\pi^2}[1-(-1)^n], \\ B_n &= 2\int_0^{1/2} x \sin n\pi x dx + 2\int_{1/2}^1 (1-x)\sin \pi x\ dx = \\ &= \frac{2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}}{n^2\pi^2} + \frac{2\sin\frac{n\pi}{2} + n\pi\cos\frac{n\pi}{2}}{n^2\pi^2} = \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2} \end{aligned}$$

Hence, the solution $u(t,x)$ of the gitar string problem is

$$u(t,x) = \sum_{n=1}^{\infty} \frac{2sin(n\pi x)}{kn^2\pi^2}((1-(-1)^n)\sin(3n\pi t) + \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}\cos(3n\pi t),$$

for $0 \le x \le 1$ and $t \ge 0$.

3.3 Exercises

Example 3.1 *Solve the following initial value problems:*

1.
$$u_{tt} = 4u_{xx}, \qquad t > 0, \qquad -\infty < x < \infty,$$
$$u(0,x) = \cos 3x, \quad u_t(0,x) = x.$$

2.
$$u_{tt} = u_{xx}, \qquad t > 0, \qquad -\infty < x < \infty,$$
$$u(0,x) = \sin 3x, \quad u_t(0,x) = \cos 3x.$$

3.
$$u_{tt} = u_{xx}, \qquad t > 0, \quad -\infty < x < \infty,$$
$$u(0,x) = \begin{cases} \sin 2x, & -\pi \le x \le \pi, \\ 0, & |x| > \pi, \end{cases} \qquad u_t(0,x) = 0,$$

Example 3.2 *For the following equation:*

$$yu_{tt} - 16xu_{xx} = 0, \qquad t > 0, \quad x > 0, \tag{3.24}$$

(a) Determine type of the equation (3.24)

(b) Find characteristic curves of the equation (3.24).

(c) Transform equation (3.24) into canonical form.

Example 3.3 *For the following equation:*

$$u_{xx} + 3u_{xy} + 2u_{yy} = 0, \qquad -\infty < x, y\infty. \tag{3.25}$$

(a) Determine type of the equation (3.25)

(b) Find characteristic curves of the equation (3.25).

(c) Transform equation (3.25) into its canonical form.

Example 3.4 *For the following equation:*

$$u_{tt} - 4u_{tx} + 4u_{xx} - u_t = 0, \qquad t\ geq0, \quad -\infty < x < \infty. \tag{3.26}$$

(a) Determine the type of the equation (3.26)

(b) Find the characteristic curves of the equation (3.26).

(c) Transform the equation (3.26) into its canonical form.

Example 3.5 *Solve the initial value problem by the D'Alembert method.*

$$\begin{aligned} &u_{tt} - 9u_{xx} = 0, \quad t \geq 0, && \infty < x < \infty \\ &u(0, x) = \cos 4x, \quad u_t(0, x) = \sin 4x, && -\infty < x < \infty. \end{aligned} \tag{3.27}$$

Example 3.6 *Solve the initial boundary value problem by the method of separation of variables.*

$$\begin{aligned} &u_{tt} - 9u_{xx} = 0, && t \geq 0, && 0 \leq x \leq 4 \\ &u(0, x) = x(4 - x), && u_t(0, x) = 1, && 0 \leq x \leq 2, \\ &u(t, 0) = 0, && u(t, 2) = 0, && t \geq 0. \end{aligned} \tag{3.28}$$

Example 3.7 *Consider the telegraphic equation*

$$u_{tt} + u_t + u = c^2 u_{xx}, \qquad t \geq 0, \quad 0 \leq x \leq L.$$

Find the solution $u(t,x)$ of the telegraphic equation which satisfies the initial condition

$$u(0,x) = x(L-x), \qquad u_t(0,x) = 0, \qquad 0 \leq x \leq L,$$

and the homogeneous boundary value conditions

$$u(t,0) = 0, \qquad u(t,L) = 0, \qquad t \geq 0.$$

Hint: Apply the method of separation of the variables t and x.

Chapter 4

Parabolic Equations

4.1 Initial Boundary Value Problem

We shall consider the heat equation

$$u_t = k^2 u_{xx} + f(t,x), \qquad t \geq 0, \qquad 0 \leq x \leq L, \tag{4.1}$$

with the initial condition

$$u(0,x) = \phi_0(x), \qquad 0 \leq x \leq L, \tag{4.2}$$

and with the boundary conditions

$$u(t,0) = \psi_0(t), \qquad u(t,L) = \psi_L(t), \qquad t \geq 0. \tag{4.3}$$

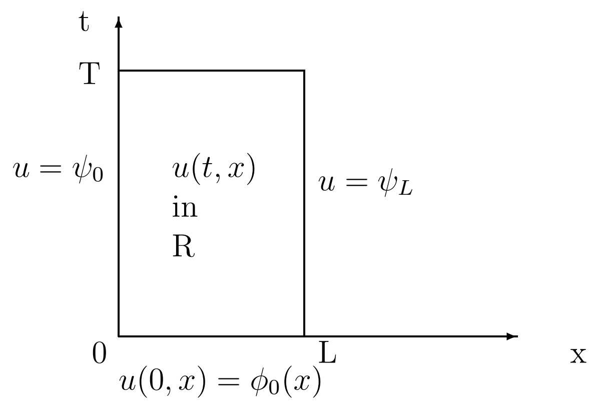
Here, k^2 is a constant and the given functions $f(t,x), \phi_0(x), \psi_0(t), \psi_L(t)$ are continuous for $0 \leq x \leq L,\ t \geq 0$.
Let us establish some of the properties of the solution $u(t,x)$. Firstly, we shall state the weak maximum principle for the heat equation

$$u_t = k^2 u_{xx} \tag{4.4}$$

in the closed rectangle $R = \{(t,x) : 0 \leq x \leq L,\ 0 \leq t \leq T\}$, with the boundary

$$\partial R = \begin{cases} (t,x): \ (0 \leq x \leq L) & \text{when} \quad t = 0 \\ (t,x): \ 0 \leq t < T & \text{when} \quad x = 0 \ \text{ or } \ x = L \end{cases}$$

Let us note that the points on the interval (T,x), when $0 \leq x \leq L$, are not included in the boundary ∂R.



The initial boundary conditions at ∂R

The maximum principle 4.1 *Let $u(t,x)$ be a solution of the heat equation (4.4) in the rectangle R. Then $u(t,x)$ assumes its maximum value on whole closed rectangle R at a point on the boundary ∂R. Also, $u(t,x)$ attains its minimum at a point on the boundary ∂R of R.*

Proof. We know that $u(t,x)$ attains its maximum M at the closed rectangle $\overline{R}$. Also, we know that $u(t,x)$ attains its maximum $M_{\partial R}$ at the closed boundary ∂R. To proof the thesis of the maximum principle, we shall show that

$$M = M_{\partial R}.$$

That is, the maximum on the boundary does not exceeds the maximum on the closed rectangle, so that $M_{\partial R} \leq M$. Suppose that $M - M_{\partial R} = \epsilon$, then we choose a point $(t_0, x_0) \in R$ interior to R, such that $u(t_0, x_0) = M$. Since $\epsilon > 0$ and (t_0, x_0) is not on the boundary ∂R, therefore $0 < x_0 < L$ and $0 < t_0 < T$. Define the auxiliary function

$$w(t,x) = u(t,x) + \frac{\epsilon}{4L^2}(x - x_0)^2.$$

Then, consider $w(t,x)$ at points on ∂R. We note that

$$\begin{aligned} w(0,x) &= u(0,x) + \frac{\epsilon}{4L^2}(x-x_0)^2 \leq M_{\partial R} + \frac{\epsilon}{4L^2}(x-x_0)^2 \\ &\leq M - \epsilon + \frac{\epsilon}{4L^2}L^2 = M - \frac{3\epsilon}{4} < M. \end{aligned}$$

In similar way, we arrive at the inequalities

$$w(t,0) < M, \qquad \text{and} \qquad w(t,L) < M, \qquad 0 \leq t \leq T.$$

Indeed, we have

$$\begin{aligned} w(t,0) &= u(t,0) + \frac{\epsilon}{4L^2}(0-x_0)^2 \le M_{\partial R} + \frac{\epsilon}{4L^2}(0-x_0)^2 \\ &\le M - \epsilon + \frac{\epsilon}{4L^2}L^2 = M - \frac{3\epsilon}{4} < M. \end{aligned}$$

and

$$\begin{aligned} w(t,L) &= u(t,L) + \frac{\epsilon}{4L^2}(L-x_0)^2 \le M_{\partial R} + \frac{\epsilon}{4L^2}(L-x_0)^2 \\ &\le M - \epsilon + \frac{\epsilon}{4L^2}L^2 = M - \frac{3\epsilon}{4} < M. \end{aligned}$$

But $w(t_0,x_0) = u(t_0,x_0) = M$. Therefore the maximum of $w(t,x)$ on R is at least M and it is attained at a point $(t_1,x_1) \in R$, not on the boundary ∂R. Because $0 < x_1 < L,\ 0 < t_1 < T,$ then

$$w_t(t_1,x_1) = 0, \qquad w_{xx}(t_1,x_1) \le 0. \tag{4.5}$$

Hence

$$w_t(t_1,x_1) - k^2 w_{xx}(t_1,x_1) \ge 0. \tag{4.6}$$

But

$$w_t(t_1,x_1) - k^2 w_{xx}(t_1,x_1) = u_t(t_1,x_1) - k^2 u_{xx}(t_1,x_1) - k^2\frac{k^2\epsilon}{2L^2} = -\frac{k^2\epsilon}{2L} < 0. \tag{4.7}$$

Thus, we have arrived at the contradiction, the inequality (4.6) against the inequality (4.7). Therefore, $u(t,x)$ attains its maximum value on the boundary ∂R of the rectangle R.
Similarly, we can prove for minimum of $u(t,x)$, taking $-u(t,x)$ instead of $u(t,x)$.
Then, we conclude that $M = M_{\partial R}$. End of the proof.
Conclusion. From the weak maximum principle, it follows that every solution $u(t,x)$ of the initial boundary value problem (4.1), (4.2),(4.3) satisfies the inequality

$$|u(t,x)| \le \max_{(t,x)\in\partial R} |u(t,x)|, \qquad t \ge 0, \quad 0 \le x \le L. \tag{4.8}$$

As a consequence of the maximum principle, we can state the following theorems

Theorem 4.1 *(***Uniqueness***)* *The initial boundary value problem (4.1),(4.2) and (4.3) has at most one continuous solution.*

Proof. Assume that there are two solutions $u_1(t,x)$ and $u_2(t,x)$. Then, it is easy to show that the difference $w(t,x) = u_1(t,x) - u_2(t,x)$ is the solution of the homogeneous heat equation with the homogeneous initial boundary conditions. By the maximum principle, we conclude that such a homogeneous initial boundary value problem has only trivial solution, that is $w(t,x) \equiv 0$ for all $(t,x) \in \overline{R}$. Indeed, because $w(t,x)$ attains its total maximum on $\overline{R}$ at the boundary ∂R, not greater than zero, therefore $w(t,x) \leq 0$ for all $(t,x) \in \overline{R}$. But, also $w(t,x)$ attains its total minim on $\overline{R}$ at the boundary ∂R. So that $w(t,x) \geq 0$ for $(t,x) \in \overline{R}$. Hence $w(t,x) \equiv 0$ for all $(t,x) \in \overline{R}$. End of the proof.

Theorem 4.2 *Let $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ be two solutions of the two initial boundary value problems*

$$\begin{aligned} &u^{(1)}_t = k^2 u^{(1)}_{xx} + f(t,x), && 0 \leq x \leq L, && t \geq 0, \\ &u^{(1)}(0,x) = \phi^{(1)}_0(x), && 0 \leq x \leq L, && \\ &u^{(1)}(t,0) = \psi^{(1)}_0(t), && u^{(1)}(t,L) = \psi^{(1)}_L(t), && t \geq 0, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &u^{(2)}_t = k^2 u^{(2)}_{xx} + f(t,x), && 0 \leq x \leq L, && t \geq 0, \\ &u^{(2)}(0,x) = \phi^{(2)}_0(x), && 0 \leq x \leq L, && \\ &u^{(2)}(t,0) = \psi^{(2)}_0(t), && u^{(2)}(t,L) = \psi^{(1)}_L(t), && t \geq 0, \end{aligned} \tag{4.10}$$

for the same sours of energy $f(t,x)$. Suppose that the distance of the initial boundary conditions is less than $\epsilon > 0$, so that

$$|\phi^{(1)}_0(x) - \phi^{(2)}_0(x)| < \epsilon, \quad |\psi^{(1)}_0(t) - \psi^{(2)}_0(t)| < \epsilon, \quad |\psi^{(1)}_L(t) - \psi^{(2)}_L(t)| < \epsilon, \tag{4.11}$$

Then, the inequality

$$|u^{(1)}(t,x) - u^{(2)}(t,x)| < \epsilon,$$

holds for all $t \geq 0$, and $0 \leq x \leq L$.

Proof. Let us note that the difference

$$v(t,x) = u^{(1)}(t,x) - u^{(2)}(t,x)$$

is the solution of the heat equation

$$v_t = k^2 v_{xx}, \qquad t \geq 0, \qquad 0 \leq x \leq L,$$

which satisfies the initial condition

$$v(0,x) = \phi^{(1)}_0(x) - \phi^{(2)}_0(x), \qquad 0 \leq x \leq L,$$

and the boundary conditions

$$v(t,0) = \psi_0^{(1)}(t) - \psi_0^{(2)}(t), \qquad v(t,L) = \psi_L^{(1)}(t) - \psi_L^{(2)}(t), \qquad t \geq 0.$$

By the assumption, $|v(0,x)| < \epsilon$, $|v(t,0)| < \epsilon$, $|v(t,L)| < \epsilon$. Then, by the maximum principle

$$|v(t,x)| < \epsilon,$$

for all $t \geq 0$ and $0\ leqx \leq L$.
Question 1. Consider the following initial value problem:

$$u_t = k^2 u_{xx}, \qquad -\infty < x < \infty, \quad t \geq 0,$$

$$u(0,x) = \phi_0(x), \quad -\infty < x < \infty,$$

Assume that the given function $\phi_0(x)$ is continuous and bounded for all $x \in (-\infty, \infty)$ and $u(t,x) \to 0$ when $x \to \mp\infty$. Using the maximum principle show that

$$|u(t,x)| \leq \max_{-\infty<x<\infty} |\phi_0(x)|.$$

for all $t \geq 0$ and $-\infty < x < \infty$.
Solution. Let $x \in [-a, a]$ for a positive $a > 0$. Then, by the weak maximum principle (see (4.8))

$$|u(t,x)| \leq \max_{(t,x)\in\partial R} |u(t,x)|, \qquad t \geq 0, \quad -a \leq x \leq a.$$

where $\partial R = \{(0,x), (t,-a)), (t,a)\}$.
By the assumption $u(t,x) \to 0$, when $x \to \mp\infty$, so that, for sufficiently large a, we have

$$|u(t,x)| \leq \max_{x\in[-a,a]} |\phi_0(x)|,$$

for all $t \geq 0$ and $|x| \geq a$. Therefore, the inequality

$$|u(t,x)| \leq \max_{x\in[-\infty,\infty]} |\phi_0(x)|,$$

for all $t \geq 0$ and $-\infty < x < \infty$.

4.2 Solution by Separation of Variables

. Let us consider the following initial boundary value problem

$$\begin{aligned} &u_t = k^2 u_{xx}, && t \geq 0, \qquad 0 \leq x \leq L, \\ &u(0,x) = \phi_0(x), && 0 \leq x \leq L, \\ &u(t,0) = 0, && u(t,L) = 0, \quad t \geq 0. \end{aligned} \tag{4.12}$$

Substituting $u(t,x) = T(t)X(x)$, to the heat equation, we obtain

$$\frac{X''}{X} = \frac{T'}{k^2 T} = -\lambda,$$

where λ is the separation constant. Hence, we get the equations

$$X'' + \lambda X = 0, \qquad T' + \lambda k^2 T = 0. \tag{4.13}$$

Solving the boundary value problem for the ordinary differential equation

$$X'' + \lambda X = 0, \qquad X(0) = X(L) = 0, \tag{4.14}$$

we arrive at the solution

$$X_n(x) = \sin\frac{n\pi x}{L}, \qquad \lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, ... \tag{4.15}$$

Now, the equation for $T(t)$ becomes

$$T' + \frac{k^2 n^2 \pi^2}{L^2} T = 0, \qquad n = 1, 2, ...$$

with the solution

$$T_n(t) = e^{\frac{-k^2 n^2 \pi^2 t}{L^2}}, \qquad n = 1, 2, ... \tag{4.16}$$

Hence, we find the solution of the heat equation

$$u_n(t,x) = T_n(t)X_n(t) = e^{\frac{-k^2 n^2 \pi^2 t}{L^2}} \sin\frac{n\pi x}{L}, \qquad n = 1, 2, ... \tag{4.17}$$

which satisfies the homogeneous boundary conditions. The function

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{k^2 \pi^2 n^2 t}{L^2}} \sin\frac{n\pi x}{L}, \tag{4.18}$$

is also the solution of the heat equation and satisfies the homogeneous boundary conditions for any choice of the coefficients $B_n,\ n = 1, 2, ...$

In order to determine the coefficients $B_n,\ n = 1, 2, ...$, we expand in the Fourier series of sines the function $\phi_0(x)$ given in the initial condition. Then, we have

$$u(0,x) = \phi_0(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Having Fourier series of the initial function

$$\phi_0(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad B_n = \frac{2}{L}\int_0^L \phi_0(\xi) \sin\frac{n\pi\xi}{L} d\xi,$$

we arrive at the solution of the initial boundary value problem

$$u(t,x) = \frac{2}{L}\sum_{n=1}^{\infty}[\int_0^L \phi_0(\xi)\sin\frac{n\pi\xi}{L}d\xi]e^{\dfrac{-n^2\pi^2k^2t}{L^2}}\sin\frac{n\pi x}{L} \tag{4.19}$$

Example 1. Solve the following initial boundary value problem:

$$\begin{aligned} &u_t = k^2u_{xx}, && t \geq 0, \quad 0 \leq x \leq 1, \\ &u(t,0) = 0, && u(t,1) = 0, \quad t \geq 0, \\ &u(0,x) = \phi_0(x) = \sin\pi x + \frac{1}{2}\sin 3\pi x, && \end{aligned} \tag{4.20}$$

Solution. We note that the coefficients of the Fourier series of the initial value function

$$\phi_0(x) = \sin\pi x + \frac{1}{2}\sin 3\pi x,$$

are $B_1 = 1,\ B_2 = 0,\ B_3 = \frac{1}{2},\ B_4 = B_5 = ... = 0.$
Therefore, the solution is

$$u(t,x) = e^{-\pi^2k^2t}\sin\pi x + \frac{1}{2}e^{-9\pi^2k^2t}\sin 3\pi x.$$

Example 2. Solve the following initial boundary value problem by the method of separation of variables

$$\begin{aligned} &u_t = k^2u_{xx}, && t \geq 0, \quad 0 \leq x \leq L, \\ &u(t,0) = 0, && u(t,L) = 0, \quad t \geq 0, \\ &u(0,x) = \phi_0(x) = \begin{cases} x, & 0 \leq x \leq \frac{L}{2}, \\ L - x, & \frac{L}{2} \leq x \leq L \end{cases} && \end{aligned} \tag{4.21}$$

Solution. We compute the coefficients $B_n,\ n = 1,2,...$ of the Fourier series of the initial value function $\phi_0(x)$

$$B_n = \frac{2}{L}\int_0^L \phi_0(\xi)\sin\frac{n\pi\xi}{L}d\xi = 2L\frac{(-1)^n - \sin\frac{n\pi}{2}}{2n\pi} + \frac{2\sin\frac{n\pi}{2}}{n^2\pi^2}]$$

Hence, by the formula (4.19), we get the solution

$$u(t,x) = 2L\sum_{n=1}^{\infty}[\frac{(-1)^n - \sin\frac{n\pi}{2}}{2n\pi} + \frac{2\sin\frac{n\pi}{2}}{n^2\pi^2}]Exp[-\frac{-\pi^2n^2k^2t}{L}]\sin\frac{\pi nx}{L} \tag{4.22}$$

4.3 Transformation of Non-homogeneous Boundary Conditions to Homogeneous

Let us consider the heat equation with constant temperature at the end of a rod. So, we consider the following initial boundary value problem:

$$\begin{array}{lll} u_t = k^2 u_{xx}, & t \geq 0, & 0 \leq x \leq L, \\ u(t,0) = \psi_0, & u(t,L) = \psi_L, & t \geq 0, \\ u(0,x) = \phi_0(x), & & \end{array} \tag{4.23}$$

where this time ψ_0, ψ_L are constants.
In order to transform the non-homogeneous boundary conditions to homogeneous ones, we introduce the new unknown function $v(t,x)$ by the formula

$$u(t,x) = v(t,x) + \psi_0 + \frac{x}{L}(\psi_L - \psi_0).$$

Clearly, the unknown $v(t,x)$ satisfies the homogeneous boundary conditions $v(t,0) = v(t,L) = 0$. So that $v(t,x)$ is the solution of initial boundary problem

$$\begin{array}{lll} v_t = k^2 v_{xx}, & t \geq 0, & 0 \leq x \leq L, \\ v(t,0) = 0, & u(t,L) = 0, & t \geq 0, \\ v(0,x) = \phi_0(x) - [\psi_0 + \frac{x}{L}(\psi_L - \psi_0)], & 0 \leq x \leq L. & \end{array} \tag{4.24}$$

By the formula (4.22), we find the solution

$$v(t,x) = \frac{2}{L}\sum_{n=1}^{\infty}[\int_0^L \overline{\phi}_0(\xi)\sin\frac{n\pi\xi}{L}d\xi]e^{\frac{-n^2\pi^2k^2t}{L^2}}\sin\frac{n\pi x}{L}, \tag{4.25}$$

where $\overline{\phi}_0(\xi) = \phi_0(x) - [\psi_0 + \frac{x}{L}(\psi_L - \psi_0)]$. Finally, in terms of original unknown

$$u(t,x) = v(t,x) + [\psi_0 + \frac{x}{L}(\psi_L - \psi_0)]. \tag{4.26}$$

Example 3. Solve the following initial boundary value problem:

$$\begin{array}{lll} u_t = 4u_{xx}, & t \geq 0, & 0 \leq x \leq 2, \\ u(t,0) = 1, & u(t,2) = 4, & t \geq 0, \end{array} \tag{4.27}$$

$$u(0,x) = 1, \quad 0 \leq x \leq 2. \tag{4.28}$$

Solution. By introducing the new unknown function $v(t,x)$ by the formula

$$\begin{aligned} u(t,x) = v(t,x) + \psi_0 + \tfrac{x}{L}(\psi_L - \psi_0) &= v(t,x) + 1 + \tfrac{x}{2}(4-1) \\ &= v(t,x) + 1 + \tfrac{3x}{2}, \end{aligned} \tag{4.29}$$

we note that

$$v_t(t,x) = u_t(t,x), \qquad v_{xx} = u_{xx}(t,x),$$

Therefore $v(t,x)$ is the solution of the heat equation. So that

$$v_t = k^2 v_{xx}, \qquad t \geq 0, \quad 0 \leq x \leq L.$$

and

$$u(t,0) = v(t,0) + 1 = 1,$$

$$u(t,2) = v(t,2) + 1 + \tfrac{2}{2}(4-1) = v(t,2) + 4 = 4.$$

Hence, we obtain

$$v(t,0) = 0, \qquad v(t,2) = 0, \qquad t \geq 0.$$

Then, the new unknown function $v(t,x)$ satisfies the initial condition

$$\begin{aligned} v(0,x) = \overline{\phi}_0(x) &= \phi_0(x) - [\psi_0(0) + \frac{x}{2}(\psi_L(t) - \psi_0(t)) \\ &= 1 - [1 + \frac{x}{2}(4-1)] = -\frac{3x}{2}. \end{aligned}$$

Hence, by the formula (4.19), we obtain the solution

$$\begin{aligned} v(t,x) &= \frac{2}{L}\sum_{n=1}^{\infty}[\int_0^L \overline{\phi}_0(\xi)\sin\frac{n\pi\xi}{L}d\xi]e^{\dfrac{-n^2\pi^2k^2t}{L^2}}\sin\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty}[\int_0^2 -\frac{3\xi}{2}\sin\frac{n\pi\xi}{2}d\xi]e^{\dfrac{-4n^2\pi^2t}{4}}\sin\frac{n\pi x}{2} \\ &= \sum_{n=1}^{\infty}\frac{3}{n\pi}e^{-n^2\pi^2t}\sin\frac{n\pi x}{2} \end{aligned} \tag{4.30}$$

Now, coming back to the original unknown function $u(t,x)$, by the formula (4.29), we find the solution

$$u(t,x) = 1 + \frac{3x}{2} + \sum_{n=1}^{\infty}\frac{3}{n\pi}e^{-n^2\pi^2t}\sin\frac{n\pi x}{2}, \quad t \geq 0, \quad 0 \leq x \leq 2.$$

4.4 More Developed Heat Equation

Let us consider the following initial boundary value problem:

$$\begin{aligned} &u_t = k^2u_{xx} - \beta u, \quad 0 \leq x \leq L, \quad t \geq 0, \\ &u(0,x) = \phi_0(x), \quad 0 \leq x \leq L, \\ &u(t,0) = 0, \qquad u(t,L) = 0, \quad t \geq 0 \end{aligned} \tag{4.31}$$

Here, the term $-\beta u,\ \beta > 0$, represents heat flow across the lateral boundary. Let us note that by the substitution

$$u(t,x) = e^{-\beta t}v(t,x), \quad u_t = -\beta e^{-\beta t}v(t,x) + e^{-\beta t}v_t(t,x), \quad u_{xx} = e^{-\beta t}v_{xx}(t,x)$$

we can transform the initial boundary problem (4.31) into simple one for the new unknown $v(t,x)$.

$$\begin{aligned} &v_t = k^2 v_{xx}, && 0 \le x \le L, \quad t \ge 0,\\ &v(0,x) = \phi_0(x), && 0 \le x \le L,\\ &v(t,0) = 0, && v(t,L) = 0, \quad t \ge 0 \end{aligned} \tag{4.32}$$

Solving the initial boundary value problem (4.32), by the formula (4.19), we obtain the solution

$$u(t,x) = e^{-\beta t}v(t,x) = \frac{2}{L}e^{-\beta t}\sum_{n=1}^{\infty}[\int_0^L \phi_0(\xi)\sin\frac{n\pi\xi}{L}d\xi]e^{\frac{-n^2\pi^2k^2t}{L^2}}\sin\frac{n\pi x}{L} \tag{4.33}$$

Example 4. Let us consider the initial boundary value problem:

$$\begin{aligned} &u_t = u_{xx} - u, && 0 \le x \le L, \quad t \ge 0,\\ &u(0,x) = \sin\pi x \ + \ \frac{1}{2}\sin 3\pi x, && 0 \le x \le L,\\ &u(t,0) = 0, && v(t,L) = 0, \quad t \ge 0, \end{aligned} \tag{4.34}$$

Solution. We apply the substitution

$$u(t,x) = e^{-t}v(t,x),$$

to eliminate the term $-u$. Then, $v(t,x)$ satisfies the heat equation with the initial boundary conditions

$$\begin{aligned} &v_t = u_{xx}, && 0 \le x \le L, \qquad t \ge 0,\\ &v(0,x) = \sin\pi x \ + \ \frac{1}{2}\sin 3\pi x, && 0 \le x \le L,\\ &v(t,0) = 0, && 0, u(t,L) = 0, \quad t \ge 0, \end{aligned} \tag{4.35}$$

The solution of the problem (4.35) is

$$v(t,x) = e^{-\pi^2 t}\sin\pi x \ + \ \frac{1}{2}e^{-9\pi^2 t}\sin 3\pi x,$$

Hence, coming back to the original unknown, we find the solution

$$u(t,x) = e^{-t}[e^{-\pi^2 t}\sin\pi x \ + \ \frac{1}{2}e^{-9\pi^2 t}\sin 3\pi x],$$

4.5 Non-homogeneous Heat Equation

Let us consider the non homogenous heat equation with the initial boundary value conditions

$$\begin{aligned} &u_t = k^2 u_{xx} + f(t,x), \quad t \geq 0, \qquad 0 \leq x \leq L, \\ &u(0,x) = \varphi(x), \qquad 0 \leq x \leq L, \\ &u(t,0) = 0, \qquad u(t,L) = 0, \quad t \geq 0. \end{aligned} \tag{4.36}$$

In the previous section (see (4.18), we have found the solution

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\dfrac{k^2\pi^2 n^2 t}{L^2}} \sin\frac{n\pi x}{L}. \tag{4.37}$$

when $f(t,x) \equiv 0,\ \psi_0(t) \equiv 0,\ \psi_l(t) \equiv 0$. Now, we shall find the solution $u(t,x)$ of the non homogeneous heat equation, when $f(t,x) \neq 0$.
Assume that the given function as the heat sourse $f(t,x)$ possesses the following series presentation:

$$f(t,x) = f_1(t)\sin\frac{\pi x}{L} + f_2(t)\sin\frac{2\pi x}{L} + ... + f_n(t)\sin\frac{n\pi x}{L} + ... \tag{4.38}$$

In order to find the coefficients $f_n(t),\ n = 1,2,...;$ we multiply both sides of (4.38) by $\sin\dfrac{m\pi x}{L}$, and integrate from zero to L with respect to the variable x. Then, we obtain

$$\int_0^L f(t,x)\sin\frac{n\pi x}{L}\,dx = \sum_{n=1}^{\infty} f_n(t)\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{L}{2}f_m(t). \tag{4.39}$$

Hence, by the orthogonality of the sequence $\{\sin\dfrac{n\pi x}{L},\ n = 1,,2,...;\}$ we find the coefficients

$$f_n(t) = \frac{2}{L}\int_0^L f(t,x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1,2,...; \tag{4.40}$$

Replacing the heat sourse function $f(t,x)$ by its decomposition (4.38). we find the solution

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L} \tag{4.41}$$

of the initial boundary value problem

$$\begin{aligned} &u_t = k^2 u_{xx} + \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}, \quad t \geq 0, \qquad 0 \leq x \leq L, \\ &u(0,x) = \varphi(x), \qquad 0 \leq x \leq L, \\ &u(t,0) = 0, \qquad u(t,L) = 0, \quad t \geq 0. \end{aligned} \tag{4.42}$$

where the functions $T_n(t),\ n = 1, 2, ...;$ are to be determined.
By substitution (4.41) to the equations (4.42), we have the following equations:

$$\sum_{n=1}^{\infty} T_n^{'}(t)\sin\frac{n\pi x}{L} = -k^2\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^2}T_n(t)\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}$$
$$\sum_{n=1}^{\infty} T_n(t)\sin 0 = 0, \qquad \sum_{n=1}^{\infty} T_n(t)\sin n\pi = 0, \qquad t \geq 0, \tag{4.43}$$
$$\sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi x}{L} = \varphi(x), \qquad 0 \leq x \leq L.$$

Hence

$$\sum_{n=1}^{\infty}[T^{'}(t) + \frac{k^2n^2\pi 2}{L^2}T_n(t) - f_n(t)]\sin\frac{n\pi x}{L} = 0,$$
$$\sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi x}{L} = \varphi(x). \tag{4.44}$$

Then,we find the coefficients $T_n(t),\ n = 1, 2, ...;$ solving the ordinary differential equations

$$T_n^{'}(t) + \frac{k^2\pi^2n^2}{L^2}T_n(t) = f_n(t), \qquad t \geq 0, \quad n = 1, 2, ...,$$
$$T_n(0) = \frac{2}{L}\int_0^L \varphi(x)\sin\frac{n\pi x}{L} = B_n. \tag{4.45}$$

By integrating factor method, we find the solution

$$T_n(t) = B_n e^{-\frac{k^2\pi^2n^2t}{L^2} + \int_0^t e^{-\frac{k^2\pi^2n^2}{L^2}(t-\tau)}} f_n(\tau)\, d\tau. \tag{4.46}$$

Finally, the solution of the initial boundary value problem (4.36) is given by the following formula:

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{k^2\pi^2n^2t}{L^2}}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\int_0^t e^{-\frac{k^2\pi^2n^2(t-\tau)}{L^2}} f_n(\tau)\, d\tau. \tag{4.47}$$

Example 4.1 *Solve the following initial boundary value problem:*

$$u_t = u_{xx} + \sin\pi x + \sin 2\pi x, \quad 0 \leq x \leq 1, \quad t \geq 0,$$
$$u(0,x) = \sin\pi x, \qquad 0 \leq x \leq 1, \tag{4.48}$$
$$u(t,0) = 0, \qquad u(t,1) = 0, \qquad t \geq 0.$$

Solution. We shall find the solution $u(t,x)$ in the series form

$$u(t,x)=\sum_{n=1}^{\infty}T_n(t)\sin n\pi x, \qquad 0\le x\le 1, \quad t\ge 0,$$

where the coefficients $T_n(t),\ n=1,2,...;$ are determined by the ordinary differential equation

$$\begin{aligned}&T^{'}(t)+n^2\pi^2T_n(t)=\int_0^1[\sin\pi s+\sin 2\pi s]\sin n\pi x\; ds=\frac{1}{2}f_n(t), \qquad 0\le x\le 1, \quad t\ge 0,\\ &T_n(0)=2\int_0^1\sin\pi s\;\sin n\pi s\; ds=B_n.\end{aligned} \tag{4.49}$$

We compute

$$f_n(t)=2\int_0^1[\sin\pi s+\sin 2\pi s]\sin n\pi x\; ds=\begin{cases}1, & n=1,2\\ 0, & otherewise,\end{cases}$$

$$B_n=2\int_0^1\sin\pi s\;\sin n\pi s\; ds\begin{cases}1, & n=1,\\ 0, & otherewise\end{cases}$$

Solving the ordinary differential equation (4.49) for $n=1,2,...;$ we find

$$\begin{aligned}&n=1,\ \ T_1^{'}(t)+\pi^2T_1(t)=1, \qquad T_1(0)=1,\ \ T_1(t)=\frac{\pi^2+e^{\pi^2t}-1}{\pi^2e^{\pi^2}}\\ &n=2,\ \ T_2^{'}(t)+4\pi^2T_2(t)=1, \qquad T_2(0)=0,\ \ T_2(t)=\frac{1-e^{-4\pi^2t}}{4\pi^2},\\ &n\ge 3,\ \ T_n^{'}(t)+n^2\pi^2T_n(t)=0,\ \ T_1(0)=0,\ \ T_n(t)=0.\end{aligned} \tag{4.50}$$

Finally, we obtain the solution

$$\begin{aligned}u(t,x)=\sum_{n=1}^{\infty}T_n(t)\sin n\pi x &= T_1(t)\sin\pi x+T_2(t)\sin 2\pi x\\ &= \frac{\pi^2+e^{\pi^2t}-1}{\pi^2e^{\pi^2}}\sin\pi x+\frac{1-e^{-4\pi^2t}}{4\pi^2}\sin 2\pi x.\end{aligned}$$

4.6 Fundamental Solution for the Heat Equation

Let us consider the heat equation

$$u_t=u_{xx}, \tag{4.51}$$

The fundamental solution of the heat equation is given by the formula

$$U(x-\xi,t-\eta)=\frac{1}{\sqrt{t-\eta}}e^{-\dfrac{(x-\xi)^2}{4(t-\eta)}}, \tag{4.52}$$

Let us note that the fundamental solution is the function of two points $P=(t,x)$ and $Q=(\xi,\eta)$ given for $\eta<t$.
Also, we note that $U(t,x;\xi,\eta)$ satisfies the heat equation as the function of the variables t and x at fixed ξ and η. Indeed, we find

$$u_t=[\frac{(x-\xi)^2}{4(t-\eta)^{5/2}}-\frac{1}{2(t-\eta)^{3/2}}]e^{-\dfrac{(x-\xi)^2}{4((t-\eta)}}$$

and

$$u_{xx}=[\frac{(x-\xi)^2}{4(t-\eta)^{\frac{5}{2}}}-\frac{1}{2(t-\eta)^{3/2}}]e^{-\dfrac{(x-\xi)^2}{4((t-\eta)}}$$

So that $u_t=u_{xx}$.
Also, one can check that the fundamental solution $U(t,x,\xi,\eta)$ satisfies the conjugate heat equation

$$u_\eta+u_{\xi\xi}=0,$$

as the function of the variables ξ and η, at fixed t and x.

4.7 Fundamental Formulae

Green's Formula. Below, we present Green's formula in its simplest form for two continuosly differentiable functions $P(x_1,x_2)$ and $Q(x_1,x_2)$

$$\int_R\int\frac{\partial Q}{\partial x_1}-\frac{\partial P}{\partial x_2}]dx_1\,dx_2=\int_{\partial R}P\,dx_1+Q\,dx_2, \tag{4.53}$$

where the rectangle

$$R=\{(x_1,x_2):\ a\le x_1\le b,\quad c\le x_2\le d\}.$$

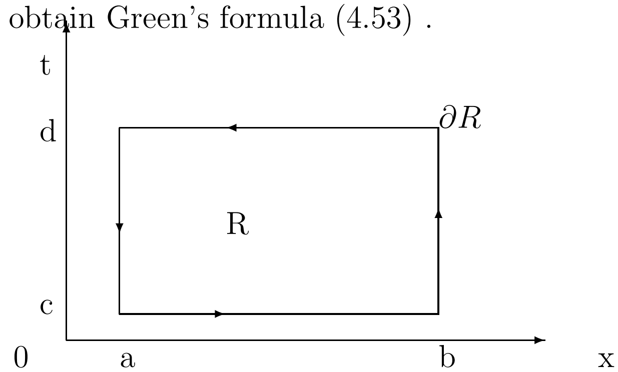
with the boundary ∂R.
Proof. We shall show that

$$\int_R\int\frac{\partial P}{\partial x_2}dx_1\,dx_2=-\int_{\partial R}P\,dx_1$$
$$\int_R\int\frac{\partial Q}{\partial x_1}dx_1\,dx_2=\int_{\partial R}Q\,dx_2$$

Indeed, we compute (see picture)

$$\begin{aligned}
\int_R\int \frac{\partial P}{\partial x_2}dx_1\,dx_2 &= \int_a^b dx_1\int_c^d \frac{\partial P}{\partial x_2}dx_2 = \int_a^b [P(x_1,d)-P(x_1,c)]dx_1\\
&= -\int_a^b P(x_1,d)dx_1 - \int_a^b P(x_1,c)dx_1 = -\int_{\partial R} P\,dx_1\\
\int_R\int \frac{\partial Q}{\partial x_1}dx_1\,dx_2 &= \int_c^d dx_2\int_a^b \frac{\partial Q}{\partial x_1}dx_1 = \int_c^d [Q(b,x_2)-Q(c,x_2)]dx_2\\
&= \int_c^d Q(b,x_2)dx_2 - \int_c^d Q(c,x_2)dx_2 = \int_{\partial R} Q\,dx_2
\end{aligned}$$

Hence, we obtain Green's formula (4.53) .



The rectangle R with the boundary ∂R

.

For $Q = u\,v$ and $P = 0$, from, Green's formula, by the identities

$$\begin{aligned}
\int_R\int \frac{\partial uv}{\partial x_1}dx_1\,dx_2 = \int_R\int u\frac{\partial v}{\partial x_1} + v\frac{\partial u}{\partial x_1}dx_1\,dx_2 = \int_{\partial R} uv\,ds\\
\int_R\int \frac{\partial uv}{\partial x_2}dx_1\,dx_2 = \int_R\int u\frac{\partial v}{\partial x_2} + v\frac{\partial u}{\partial x_2}dx_1\,dx_2 = \int_{\partial R} uv\,ds
\end{aligned} \tag{4.54}$$

we obtain the formula of integration by parts in two variables

$$\begin{aligned}
\int_R\int u\frac{\partial v}{\partial x_1} = \int_{\partial R} uv\,ds - \int_R\int v\frac{\partial u}{\partial x_1}dx_1\,dx_2\\
\int_R\int u\frac{\partial v}{\partial x_2} = \int_{\partial R} uv\,ds - \int_R\int v\frac{\partial u}{\partial x_2}dx_1\,dx_2
\end{aligned} \tag{4.55}$$

Here the line integral along the boundary ∂R of the rectangle R is

$$\int_{\partial R} u\,v\,ds = \int_{\partial R} u(x_1(\xi),x_2(\xi))v(x_1(\xi),x_2(\xi))\sqrt{(x_1'(\xi))^2+(x_2'(\xi))^2}d\xi$$

which becomes

$$\int_{\partial R} u\ v\ ds = \int_a^b u(x_1(\xi), c)v(x_1(\xi), c)d\xi + \int_b^d u(b, x_2(\xi))v(b, x_2(\xi))d\xi$$
$$+\int_b^c u(x_1(\xi), d)v(x_1(\xi), d)d\xi + \int_d^a u(a, x_2(\xi))v(a, x_2(\xi))d\xi$$

The first fundamental formula for the heat equation Let us note that the following identity holds:

$$vF(u) - uG(v) = \frac{\partial}{\partial x}[v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}] - \frac{\partial(uv)}{\partial t},$$

for

$$F(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}, \qquad G(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial t}$$

Integrating by parts both sides of the above identity (see formula integration by parts (4.54)), in the rectangle

$$R = \{(t, x):\ 0 \leq t \leq T,\ 0 \leq x \leq L\}$$

we obtain the first fundamental formula for the heat equation

$$\int_R [vF(u) - uG(v)]dt\ dx = \int_{\partial R} u\ v\ dx + (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})\ dt \tag{4.56}$$

Hence, if u and v satisfy the equations

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} = 0,$$

then the first fundamental formula for the heat equation becomes

$$\int_{\partial R} u\ v\ dx + (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})\ dt = 0. \tag{4.57}$$

The second fundamental formula for the heat equation Substituting into the first fundamental formula, $\eta = t - \delta,\ v(\xi, \eta) = U(t, x; \xi, \eta)$ for $\delta < \eta$, we obtain

$$\begin{aligned}\int_{\partial R} u(\xi, t-\delta)e^{-\dfrac{(x-\xi)^2}{4\delta}}\frac{d\xi}{\sqrt{\delta}} &= \int_{\partial R}[u_\xi(\xi, \eta)U(t, x; \xi, \eta) - u(\xi, \eta)U_\xi t, x; \xi, \eta)]d\eta \\ &+ \ u(\xi, \eta)U(t, x; \xi, \eta)\ d\xi.\end{aligned} \tag{4.58}$$

Hence, in the limit when $\delta- > 0$, we obtain the second fundamental formula

$$\begin{aligned}&\int_{\partial R}[u_\xi(\xi, \eta)U(t, x; \xi, \eta) - u(\xi, \eta)U_\xi(t, x; \xi, \eta)]d\eta \\ &\qquad +u(\xi, \eta)U(t, x; \xi, \eta)d\xi \quad = \begin{cases} 2\sqrt{\pi}u(t, x), & (t, x) \in R, \\ 0, & (t, x)\ out\ of\ closed\ R \end{cases}\end{aligned} \tag{4.59}$$

4.8 Exercises

Question 1. Solve the initial boundary problem:

$$\begin{aligned} &u_t = 9u_{xx}, && 0 \le x \le 4, \quad t \ge 0, \\ &u(0,x) = \sin \pi x + 2 \sin 5\pi x, && 0 \le x \le 4, \\ &u(t,0) = 1, && v(t,4) = 2, \quad t \ge 0 \end{aligned} \tag{4.60}$$

Question 2. Solve the initial boundary problem:

$$\begin{aligned} &u_t = 16u_{xx} - 3u, && 0 \le x \le 2, \quad t \ge 0, \\ &u(0,x) = \begin{cases} x, & 0 \le x \le 1, \\ 1, & 1 < x \le 2, \end{cases} \\ &u(t,0) = 1, && u(t,2) = 0, \quad t \ge 0 \end{aligned} \tag{4.61}$$

Chapter 5

Elliptic Equations

5.1 Laplace Equation

. Laplace's equation takes the following form:

1. In two variables x, y

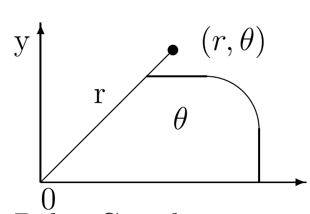
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \Delta u = u_{xx} + u_{yy} = 0, \tag{5.1}$$

   where the Laplace's operator $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

2. Laplace's equation in the polar coordinates (r, θ), $r \neq 0,\ x = r\cos\theta,\ y = r\sin\theta$,

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{5.2}$$

   where $r^2 = x^2 + y^2$ and $\theta = arctan\frac{y}{x},\ x \neq 0,\ \theta = \frac{\pi}{2},\ x = 0,$



Polar Coordinates $x = r\cos\theta,\ \ y = r\sin\theta$

Indeed, we can transform Laplace's equation from Cartesian coordinates to Laplace's equation in polar coordinates by the following computations:

$$\begin{aligned}
&x = r\cos\theta \quad y = r\sin\theta, \quad u(x,y) = u(r\cos\theta, r\sin\theta)\\
&\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\\
&\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta\\
&\frac{\partial u}{\partial \theta} = r(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta)\\
&\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta - r^2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta\\
&\qquad -r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r^2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta
\end{aligned}$$

Now, we find

$$\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= (\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta)\\
&+ \frac{1}{r}(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\\
&+ \frac{1}{r^2}(-r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta - r^2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta)\\
&+ \frac{1}{r^2}(-r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r^2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta)\\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

3. Laplace's equation in three variables x, y, z

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad \Delta u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{5.3}$$

4. Laplace's equation in the three cylindrical coordinates r, θ, z, with $r \neq 0,\ x = r\cos\theta,\ y = \sin\theta,\ z = z$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0. \tag{5.4}$$

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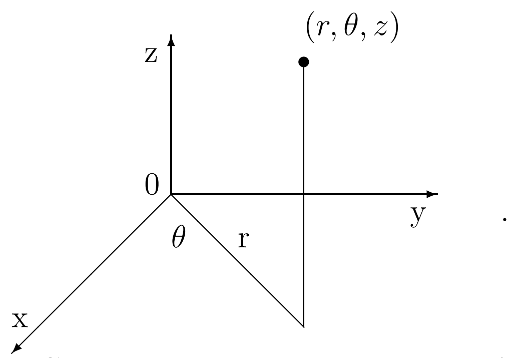


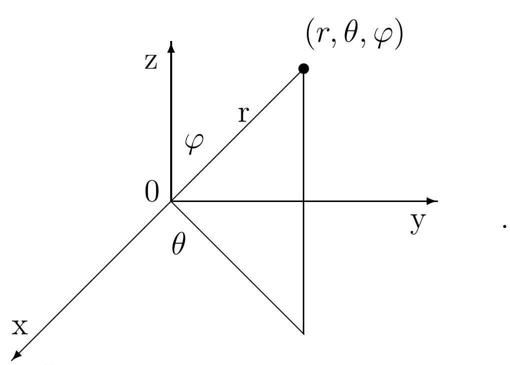
Cylindrical coordinates $x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z$

where $r^2 = x^2 + y^2, \; r \neq 0, \; \theta = arctan\dfrac{y}{x}, \; x \neq 0, \;\; z = z.$

5. Laplace's equation in the spherical coordinates r, θ, φ, with $x = r\sin\varphi \; \cos\theta, \;\; y = r\sin\varphi \; \sin\theta, \;\; z = r\cos\varphi$.

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + \frac{cot\varphi}{r^2}u_\phi + \frac{1}{r^2 sin^2\varphi}u_{\theta\theta} = 0. \tag{5.5}$$

.



Spherical coordinates $x = r\sin\varphi \; \cos\theta, \;\; y = r\sin\varphi \; \sin\theta, \;\; z = r\cos\varphi$.

where $r^2 = x^2 + y^2 + z^2, \; \cos\varphi = \dfrac{z}{r}, \;\; \tan\theta = \dfrac{y}{x}$

5.2 Boundary Value Problems for Laplace Equation

The following three types of the boundary value problems are considered:

1. **Dirichlet boundary value problem** Find the solution $u(x, y)$ of Laplace equation

$$u_{xx} + u_{yy} = 0, \qquad (x, y) \in \Omega \tag{5.6}$$

in the domain Ω, which satisfies the Dirichlet's condition (BDC)

$$u(x,y) = \phi(x,y), \qquad (x,y) \in \partial\Omega. \tag{5.7}$$

Throughout this chapter, we shall denote by Ω a bounded domain with the boundary $\partial\Omega$. Here $\phi(x,y)$ is a given function on the boundary $\partial\Omega$ of Ω.

2. **Neumann boundary problem** Find the solution $u(x,y)$ of Laplace equation

$$u_{xx} + u_{yy} = 0, \qquad (x,y) \in \Omega$$

in the domain Ω, which satisfies the Neumann's boundary condition (BNC)

$$\frac{\partial u(x,y)}{\partial n} = \psi(x,y), \qquad (x,y) \in \partial\Omega,$$

where $\dfrac{du}{dn}$ denotes normal derivative internal to the boundary $\partial\Omega$ of the domain Ω. Here $\psi(x,y)$ is a given function on the boundary $\partial\Omega$.

3. **Third kind boundary problem** Find the solution $u(x,y)$ of Laplace equation

$$u_{xx} + u_{yy} = 0, \qquad (x,y) \in \Omega$$

in the domain Ω, which satisfies the third kind boundary condition

$$A(x,y)\frac{\partial u(x,y)}{\partial n} + B(x,t)u = \phi(x,y), \qquad (x,y) \in \partial\Omega,$$

where $A^2(x,y) + B^2(x,y) > 0$ and $\mu(x,y)$ are given functions on the boundary $\partial\Omega$.

5.3 The Maximum Principle for Laplace Equation

Every solution of Laplace's equation is called harmonic function. Below, we shall give some of properties of the harmonic functions.

1. **The maximum principle.** Let $u(x,y)$ be a continuous function in the bounded and closed domain $\overline{\Omega}$. If $u(x,y)$ is a harmonic function in the domain Ω, then the function $u(x,y)$ attains its maximum and minimum values on the boundary $\partial\Omega$ of Ω.
Proof. Firstly, we note that $u(x,y)$ attains its maximum in the bounded and closed domain $\overline{\Omega}$ as a continuous function. By contradiction to the thesis, assume that the maximum value of $u(x,y)$ is not on the boundary $\partial\Omega$. Then, the maximum value of $u(x,y)$ is at some interior point

$(x_0, y_0) \in \Omega$, say $M = u(x_0, y_0) > M_b$, where M_b is maximum of $u(x, y)$ on the boundary $\partial\Omega$. Let us introduce the auxiliary function

$$v(x, y) = u(x, y) + \epsilon[(x - x_0)^2 + (y - y_0)^2],$$

for some $\epsilon > 0$. Then $v(x_0, y_0) = u(x_0, y_0) = M$, and the maximum of $v(x, y)$ on the boundary $\partial\Omega$ of Ω is equal at most $M_b + \epsilon d^2$, where d is the diameter of Ω. For sufficiently small $\epsilon > 0$, we have $M > M_b + \epsilon d$, i.e. $0 < \epsilon < (M - M_b)/d^2$. For such ϵ, the maximum of $v(x, y)$ cannot occur on the boundary of Ω, since the value M of $v(x, y)$ at (x_0, y_0) is larger than the value of $v(x, y)$ at any boundary point. There may, however, be points in Ω, where $v(x, y)$ is larger than M. Let the maximum of $v(x, y)$ be attained at (x_1, y_1), which, as we have seen, must be in Ω. At (x_1, y_1), we must have $v_{xx} \leq 0$ and $v_{yy} \leq< 0$, since the graph of $v(x, y)$ cannot be concave up in the x or y direction at (x_1, y_1). Thus, at (x_1, y_1), we have

$$v_{xx} + v_{yy} \leq 0.$$

However, by the definition of $v(x, y)$, we have

$$v_{xx} + v_{yy} + 2\epsilon + 2\epsilon = 4\epsilon > 0.$$

Here, we have used the assumption that $u(x, y)$ is harmonic on Ω. The above two inequalities contradict one the other. So that, the assumption that $u(x, y)$ attains its maximum of $u(x, y)$ at an interior point, and not on the boundary, leads to the contradiction.

In order to prove that minimum value of $u(x, y)$ is attanable at the boundary $\partial\Omega$, we repeat the proof for the maximum of $-u(x, y)$. So that, the minimum of $u(x, y)$ must be also attainable on the boundary $\partial\Omega$ of Ω.

There is also strong maximum principle for harmonic function which we present below.

The Strong maximum principle. Let $u(x, y)$ be a harmonic function on the domain Ω. Suppose that the function $u(x, y)$ attains its maximum or minimum at some interior point of Ω. Then $u(x, y)$ must be constant throughout Ω.

.

Conclusion From the maximum principle it follows that every harmonic function $u(x, y)$ satisfies the inequality

$$\min_{(x,y)\in\partial\Omega} \phi(x, y) \leq u(x, y) \leq \max_{(x,y)\in\partial\Omega} \phi(x, y),$$

Also, we have

$$|u(x, y)| \leq \max_{(x,y)\in\partial\Omega} |\phi(x, y)|, \qquad (x, y) \in \overline{\Omega}, \tag{5.8}$$

or

$$-\max_{(x,y)\in\partial\Omega} |\phi(x, y)| \leq u(x, y) \leq \max_{(x,y)\in\partial\Omega} |\phi(x, y)|, \qquad (x, y) \in \overline{\Omega},$$

where $u(x,y) = \phi(x,y)$ on the boundary $\partial\Omega$ of Ω.

.

Question 1. Consider the following boundary value problem:

$$u_{xx} + u_{yy} = 0, \qquad (x,y) \in \Omega = \{(x,y) :\ 0 < x < 2,\ 0 < y < 1\},$$
$$u(x,y) = \sin \pi x + \cos \pi y, \quad (x,y) \in \partial\Omega, \tag{5.9}$$

Find the range of the values of the solution $u(x,y)$, for $(x,y) \in \overline{\Omega}$.
Solution. By the maximum principle, the range of the values of the solution $u(x,u)$ is determined by the inequality

$$\min_{(x,y)\in\partial\Omega} \phi(x,y) \leq u(x,y) \leq \max_{(x,y)\in\partial\Omega} \phi(x,y).$$

where $\phi(x,y) = \sin \pi x + \cos \pi y$.
Now, we compute

$$\min_{(x,y)\in\partial\Omega}[\sin \pi x + \cos \pi y], \qquad \text{and} \qquad \max_{(x,y)\in\partial\Omega}[\sin \pi x + \cos \pi y]$$

Clearly, the minimum and maximum of $\phi(x,y) = \sin \pi x + \cos \pi y$ are at points when the partial derivatives are equal to zero, so that

$$\phi_x(x,y) = \pi \cos \pi x = 0, \qquad \text{and} \qquad \phi_y(x,y) = \pi \sin \pi y = 0.$$

We find two boundary points $(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 1)$ Thus, the maximum is at the point $(\frac{1}{2}, 0)$ equal

$$\max_{\partial\Omega}[\sin \pi x + \cos \pi y] = 2,$$

and the minimum is at the point $(\frac{3}{2}, 1)$ equal

$$\min_{\partial\Omega}[\sin \pi x + \cos \pi y] = -2.$$

Hence, the values of the solution $u(x,y)$ are in the range from -2 to 2, so that the following inequality holds:

$$-2 \leq u(x,y) \leq 2.$$

for all points $(x,y) \in \Omega \cup \partial\Omega$.

.

The maximum principle implies uniqueness and continuous dependence of solutions on boundary data. Namely, we shall prove the following theorems:

Theorem 5.1 (Uniqueness). *The Dirichlet boundary value problem has at most one solution.*

Proof. Assume that there are two solutions u_1 and u_2 of the Dirichlet boundary problem (5.1). Then, the function $v = u_1 - u_2$ is continuous in the closed domain $\overline{\Omega}$ and harmonic in the open domain Ω. Since $v = 0$ on $\partial\Omega$, the maximum principle implies that $v \leq 0$ and $v \geq 0$ on $\overline{\Omega}$, so that $v \equiv 0$ on $\overline{\Omega}$ and $u_1 \equiv u_2$.
Now, we shall state the theorem on continuous dependence of a harmonic function on its boundary values.

Theorem 5.2 *Let u_1 and u_2 be the solutions of the Dirichlet boundary value problems*

$$\Delta u_1 = 0 \quad in \quad \Omega, \quad u_1 = \phi_1 \quad on \quad \partial\Omega$$

$$\Delta u_2 = 0 \quad in \quad \Omega, \quad u_2 = \phi_2 \quad on \quad \partial\Omega$$

Then

$$|u_1(x, y) - u_2(x, y)| \leq M,$$

where $M = \max_{\partial\Omega} |\phi_1(x, y) - \phi_2(x, y)|$.

Proof. Let $v = u_1 - u_2$. Then, we have

$$-\max_{\partial\Omega} |\phi_1(x, y) - \phi_2(x, y)| \leq v(x, y) \leq \max_{\partial\Omega} |\phi_1(x, y) - \phi_2(x, y)|.$$

Hence, we obtain the inequality

$$|u_1(x, y) - u_2(x, y)| \leq M,$$

for all $(x, y) \in \Omega \cup \partial\Omega$.

Example 1. Suppose that $u(x, y)$ is a continuous function on the closed disk $r \leq 1$, and harmonic in the open disk $r < 1$. If

$$u(\cos\theta, \sin\theta) \leq \sin\theta + \cos 2\theta,$$

then show that

$$u(x, y) \leq y + x^2 - y^2,$$

for all $x^2 + y^2 \leq 1$.
.

Solution. Note that $v(x, y) = y + x^2 - y^2$ is a harmonic function with $v(\cos\theta, \sin\theta) = \sin\theta + \cos 2\theta$. By the assumption, $u \leq v$ on the boundary of the disk $r \leq 1$. Then, the maximum of the harmonic function $u - v$ on the boundary $r = 1$ must be less than or equal to zero. The maximum principle then implies that $u - v \leq 0$ throughout the disk.

5.4 The Maximum Principle for Poisson Equation

We shall state the maximum Principe for Poisson's equation

$$\begin{aligned} &\Delta u = f(x,y), \qquad\qquad (x,y)\in\Omega,\\ &\text{or}\\ &\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y), \quad (x,y)\in\Omega, \end{aligned} \tag{5.10}$$

where $f(x,y)$ is given continuous functions in the bounded domain Ω.
The following maximum principle holds:
Maximum principle. If $f(x,y)\geq 0$, for $(x,y)\in\Omega$, then the solution $u(x,y)$ attains non-negative maximum M at a boundary point, or if $f(x,y)\leq 0$ then $u(x,y)$ attains its non-positive minimum m at a boundary point.
This maximum principle can be proved in the same way as maximum principle for harmonic functions.
As a consequence of the above maximum principle, we can state the theorem on stability of the boundary problem for equation (5.10) with Dirichlet's boundary condition.

Theorem 5.3 *(Stability Theorem). If the given function $f(x,y)$ is continuous in the closed bounded domain $\overline{\Omega}$, then the the boundary value problem*

$$\begin{aligned} &\Delta u = f(x,y), \qquad (x,y)\in\Omega,\\ &u(x,y)=\phi(x,y) \quad (x,y)\in\Omega, \end{aligned} \tag{5.11}$$

is stable in the maximum norm, that is, the solution $u(x,y)$ satisfies the following inequality:

$$|u(x,y)| \leq \max_{X\in\partial\Omega}|u(x,y)| + M \max_{(x,y)\in\Omega}|f(x,y)|], \qquad (x,y)\in\Omega, \tag{5.12}$$

where $M=e^a-1$ for $0\leq x\leq a$.
Proof. Without any restriction, we can assume that the domain Ω is on the right side of y axis, that is, $0\leq x\leq a$. We shall prove the theorem on stability using the following lemma:

Lemma 5.1 *If a function $v(x,y)$ is a regular solution of the Poisson's equation, and if there exists a regular function $g(x,y)$, which satisfies the following conditions:*

1.

$$g(x,y)\geq \max_{(x,y)\in\partial\Omega}|v(x,y)|, \qquad (x,y)\in\Omega\cup\partial\Omega$$

2.

$$-\Delta g(x,y) \geq \max_{(x,y)\in\Omega} |f(x,y)|, \qquad (x,y) \in \Omega.$$

then

$$|v(x,y)| \leq g(x,y)$$

for all $(x,y) \in \Omega \cup \partial\Omega$*.*

Proof of lemma. In order to prove the lemma, we shall show that the functions

$$z_1(x,y) = v(x,y) - g(x,y), \qquad \text{and} \qquad z_2(x,y) = v(x,y) + g(x,y),$$

satisfy the inequalities $z_1(x,y) \leq 0$ and $z_2(x,y) \geq 0$ for all $(x,y) \in \Omega \cup \partial\Omega$.
We note that, by the definition,

$$z_1(x,y) \leq 0, \qquad z_2(x,y) \geq 0,$$

on the boundary $\partial\Omega$.
By assumption 2,

$$\begin{aligned} \Delta z_1(x,y) &= \Delta v(x,y) - \Delta g(x,y) \\ &\geq f(x,y) + \max\nolimits_\Omega |f(x,y)| \geq 0 \\ \Delta z_2(x,y) &= \Delta v(x,y) + \Delta g(x,y) \\ &\geq f(x,y) - \max\nolimits_\Omega |f(x,y)| \leq 0 \end{aligned}$$

for all $(x,y) \in \Omega$.
Hence, by the maximum principle

$$z_1(x,y) = v(x,y) - g(x,y) \leq 0, \qquad z_2(x,y) = v(x,y) + g(x,y) \geq 0,$$

hold for $(x,y) \in \Omega \cup \partial\Omega$.
Thus, we have the inequalities

$$-g(x,y) \leq v(x,y) \leq g(x,y), \qquad \text{or} \qquad |v(x,y)| \leq g(x,y)$$

for all $(x,y) \in \Omega$. End of lemma proof.
In order to prove the theorem, we assume that the domain Ω lies in the right side of x axis, that is, $x \geq 0$. This assumption can be satisfied by a linear translation of Ω in x direction. We consider the function

$$g(x,y) = \max_{(x,y)\in\partial\Omega} |u(x,y) + [e^{\overline{x}} - e^{x}] \max_{\Omega} |f(x,y)|, \qquad (x,y) \in \Omega \cup \partial\Omega.$$

where $x \leq \overline{x}$. Here, $\overline{x}$ is a fixed value on x axis.
We shall show that the function $g(x,y)$ satisfies assumptions 1 and 2 of the lemma. Indeed, we estimate

1. From the definition of $g(x, y)$, it is clear that

$$g(x, y) \geq \max_{(x,y)\in\partial\Omega} |u(x, y)|$$

   for $(x, y) \in \Omega \cup \partial\Omega$.

2. $$-\Delta g(x, y = e^{\overline{x}} \max_{\Omega} |f(x, y)| \geq \max_{\Omega} |f(x, y)|.$$

   for $0 \leq x \leq \overline{x}$.

Hence, we obtain the required inequality

$$|u(x, y)| \leq \max_{(x,y)\in\partial\Omega} |u(x, y)| + M \max_{\Omega} |f(x, y)|, \qquad (x, y) \in \overline{\Omega}.$$

where $dsM = e^a - 1$ is the upper bound of the expression $e^{\overline{x}} - e^{x}$,when $0 \leq x \leq a$.
Let us note that if the solution $u(x, y)$ satisfies the boundary condition

$$u(x, y) = \phi_0(x, y), \qquad (x, y) \in \partial\Omega,$$

then we get the following the estimate of the solution

$$|u(x, y)| \leq \max_{(x,y)\in\partial\Omega} |\phi(x, y)| + (e^a - 1) \max_{(x,y)\in\Omega} f(x, y)|], \qquad (x, y) \in \overline{\Omega}.$$

This inequality means stability of the Dirichlet boundary value problem for the Poisson's equation.
Question 1. Consider the following boundary value problem:

$$\begin{array}{ll} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin \pi x + \cos \pi y, & (x, y) \in \Omega = \{(x, y) : \ 0 < x, y < 1, \\ u(x, y) = \sin \pi xy, & (x, y) \in \partial\Omega. \end{array} \tag{5.13}$$

Show that the boundary value problem is stable and estimate the solution $u(x, y)$
Solution. We note that the Poisson's equation (5.13) satisfies the assumptions of the theorem on stability. By the thesis, we obtain the following estimate of the solution $u(x, y)$:

$$\begin{aligned} |u(x, y)| &\leq \max_{\partial\Omega} |\phi(x, y)| + M \max_{\Omega} |f(x, y)| \\ &= \max_{\partial\Omega} |\sin \pi xy| + M \max_{\Omega} |\sin \pi x + \cos \pi y)| \\ &\leq 1 + 2\ M \end{aligned}$$

Because, $0 \leq x \leq 1$ therefore, the constant $M = e - 1$ and

$$|u(x, y)| \leq 1 + 2(e - 1),$$

for all $(x, y) \in \Omega \cup \partial\Omega$.

5.5 The Maximum Principle for Helmholz-Poisson Equation

We shall state the maximum Principe for Helmholz-Poisson equation

$$\Delta u + c(x,y)u = f(x,y), \qquad (x,y) \in \Omega,$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + c(x,y)u = f(x,y), \quad (x,y) \in \Omega, \tag{5.14}$$

where $c(x,y) \leq 0$, and $f(x,y)$ are given continuous functions in the bounded domain Ω.
The following maximum principle holds:
Maximum principle. If $f(x,y) \geq 0$ and $c(x,y) \leq 0$ for $(x,y) \in \Omega$, then the solution $u(x,y)$ attains non-negative maximum M at a boundary point, or if $f(x,y) \leq 0$ and $c(x,y) \leq 0$, then $u(x,y)$ attains its non-positive minimum m at a boundary point.

As a consequence of the above maximum principle, we state the theorem on stability of the solution of Dirichlet's boundary problem for equation (5.14).

Theorem 5.4 *(Stability Theorem). If the given functions $c(x,y) \leq 0$ and $f(x,y)$ are continuous in the closed bounded domain $\overline{\Omega}$, then the solution $u(x,y)$ satisfies the following inequality:*

$$\max_{(x,y)\in\overline{\Omega}} |u(x,y)| \leq \max_{X\in\partial\Omega} |u(x,y)| + M \max_{(x,y)\in\Omega} |f(x,y)|], \tag{5.15}$$

where $M = e^a - 1$ for $0 \leq x \leq a$.
We can prove this theorem in a similar way as theorem on stability for Poisson's equation.
Let us note that if the solution $u(x,y)$ satisfies the boundary condition

$$u(x,y) = \phi_0(x,y), \qquad (x,y) \in \partial\Omega,$$

then we get the following priori estimate of the solution $u(x,y)$

$$|u(x,y)| \leq \max_{(x,y)\in\partial\Omega} |u(x,y)| + M \max_{(x,y)\in\Omega} f(x,y)|], \qquad (x,y) \in \overline{\Omega}.$$

where $M = e^a - 1$, for $0 \leq x \leq a$.
This inequality means stability of the Dirichlet boundary value problem for the Helmholz-Poisson equation.
Question 1. Consider the following boundary value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2u = \sin \pi x + \cos \pi y, \quad (x,y) \in \Omega = \{(x,y):\ 0 < x, y < 1, \\ &u(x,y) = \sin \pi xy, \qquad (x,y) \in \partial\Omega. \end{aligned} \tag{5.16}$$

Show that the boundary value problem is stable and estimate the solution $u(x, y)$

Solution. We note that the Helmholz-Poisson equation (5.16) satisfies the assumptions of the theorem on stability. Indeed, the coefficient $c(x, y) = -2 \leq 0$. By the thesis, we obtain the following estimate of the solution $u(x, y)$:

$$\begin{aligned} |u(x,y)| &\leq \max_{\partial\Omega} |\phi(x,y)| + M \max_{\Omega} |f(x,y)| \\ &= \max_{\partial\Omega} |\sin \pi xy| + M \max_{\Omega} |\sin \pi x + \cos \pi y)| \\ &\leq 1 + 2\, M \end{aligned}$$

Because, $0 \leq x \leq 1$, then the constant $M = e - 1$, and the solution $u(x, y)$ satisfies the inequality

$$|u(x, y)| \leq 1 + 2(e - 1),$$

for all $(x, y) \in \Omega \cup \partial\Omega$.

5.6 Boundary Value Problem for Laplace's Equation in a Rectangle

Let us consider the following boundary value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && (x, y) \in \Omega = \{(x, y) : \ 0 < x < L_1, \ 0 < y < L_2\}, \\ &u(x, 0) = 0, && 0 \leq x \leq L_1 \\ &u(0, y) = u(L_1, y) = 0, && 0 \leq y \leq L_2, \\ &u(x, L_2) = \varphi(x), && 0 \leq x \leq L_1. \end{aligned} \tag{5.17}$$

This boundary value problem can be solved by the method of separation of variables. Namely, let $u(x, y) = X(x)Y(y)$. Then, by substitution to Laplace's equation, we obtain

$$X''(x)Y(x) + X(x)Y''(y) = 0,$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

Hence, we have two equations

$$X''(x) + \lambda X(x) = 0, \qquad\qquad Y''(y) - \lambda Y(y) = 0.$$

From the boundary conditions

$$u(x, 0) = X(x)Y(0) = 0.$$

So that $Y(0) = 0$. Also, we have

$$u(0,y) = X(0)Y(y) = 0 \qquad \text{and} \qquad u(L_1,y) = X(L_1)Y(y) = 0$$

therefore $X(0) = X(L_1) = 0$.
Let us note the solution of the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(L_1) = 0$$

is known and the eigenvalues and eigenfunctions are given by the formulae

$$\lambda_n = \frac{n^2\pi^2}{L_1^2}$$
$$X^{(n)}(x) = \sin\frac{n\pi x}{L_1}, \qquad n = 1, 2, ...,$$

Then, we solve the corresponding equation for $Y(y)$

$$Y''(y) - \lambda_n Y(y) = 0,$$

which has the general solution

$$Y(y) = C_1 e^{\sqrt{\lambda_n}y} + C_2 e^{-\sqrt{\lambda_n}y}$$

for arbitrary constants C_1 and C_2.
By the condition $Y(0) = 0$, we find $C_1 + C_2 = 0$ and $C_2 = -C_1$, so that the solution

$$Y^{(n)}(y) = C_1(e^{\frac{n\pi y}{L_1}} - e^{-\frac{n\pi y}{L_1}}) = 2C_1 \sinh\frac{n\pi y}{L_1}, \qquad n = 1, 2, ...;$$

Now, we observe that the terms of the sequence

$$u_n(x,y) = B_n \sin\frac{n\pi x}{L_1} \sinh\frac{n\pi y}{L_1}, \qquad n = 1, 2, ...;$$

are harmonic functions and satisfy the homogeneous boundary value condition at the three sides of the rectangle Ω, that is when $y = 0$ or $x = 0$, or $x = L_1$, for arbitrary constants $B_n,\ n = 1, 2, ...,$. Then, we consider the solution in the form of the following series:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L_1} \sinh\frac{n\pi y}{L_1}, \tag{5.18}$$

where the coefficients $B_n,\ n = 1, 2, ...;$ are determined by the boundary condition

$$u(x, L_2) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L_1} \sinh\frac{n\pi L_2}{L_1} = \phi_0(x).$$

Multiplying the above identity by $\sin\dfrac{m\pi x}{L_1}$ and integrating from 0 to L_1, and using orthogonality property, we obtain the following formula for the coefficients $B_n,\ n = 1, 2, ...;$

$$B_n \sinh\frac{n\pi L_2}{L_1} = \frac{2}{L_1}\int_0^{L_1} \phi_0(s)\sin\frac{n\pi s}{L_1}\, ds$$

or

$$B_n = \frac{2}{L_1 \sinh\dfrac{n\pi L_2}{L_1}}\int_0^{L_1} \phi_0(s)\sin\frac{n\pi s}{L_1}\, ds, \qquad n = 1, 2, ...; \tag{5.19}$$

Example. Consider the following boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && (x, y) \in \Omega = \{(x, y):\ 0 < x < 2, \quad 0 < y < 4\},\\
&u(x, 0) = 0, && 0 \le x \le 2\\
&u(0, y) = u(2, y) = 0, && 0 \le y \le 4,\\
&u(x, 4) = x(2 - x), && 0 \le x \le 2.
\end{aligned} \tag{5.20}$$

Solution. By the formulae (5.18) and (5.19), we compute the coefficients

$$B_n = \frac{2}{2\sinh\dfrac{n\pi 4}{2}}\int_0^{2} s(2-s)\sin\frac{n\pi s}{2}\, ds = \frac{16(1-(-1)^n)}{n^3\pi^3\sinh 2n\pi},$$

Hence, we find the solution in the form of the following series

$$u(x, y) = \frac{16}{\pi^3}\sum_{n-1}^{\infty}\frac{(1-(-1)^n)}{n^3\sinh 2n\pi}\sin\frac{n\pi x}{2}\sinh 2n\pi y, \qquad 0 \le x \le 2,\ \ 0 \le y \le 4.$$

5.7 Boundary Value Problem for Laplace's Equation in a Disk

Let us consider the Laplace's equation in the polar coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < R, \quad -\pi < \theta < \pi, \tag{5.21}$$

with the boundary condition

$$u(R, \theta) = \phi(\theta), \qquad -\pi \le \theta \le \pi. \tag{5.22}$$

It is easy to check that the functions

$$1, \quad r^n\cos n\theta, \quad r^n\sin n\theta,$$

are harmonic in the disk with radius R and the center at the origin, for $n = 1, 2, ...$;
We shall find the solution $u(r, \theta)$ in the form of the following trigonometric series:

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta. \tag{5.23}$$

Clearly $u(r, \theta)$ given by formula (5.23) satisfies the Laplace's equation in the poplar coordinates (r, θ). In order to find the coefficients a_n and b_n, we apply the boundary condition

$$u(R, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n R^n \cos n\theta + b_n R^n \sin n\theta = \phi(\theta), \qquad -\pi \leq \theta \leq \pi.$$

So, a_n and b_n are the coefficients of the Fourier series of the given function $\phi(\theta)$ for $-\pi \leq \theta \leq \pi$.
These coefficients are given by the formulae

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(s)\, ds, \\ a_n &= \frac{1}{R^n \pi}\int_{-\pi}^{\pi} \phi(s) \cos ns\, ds, \quad b_n = \frac{1}{R^n \pi}\int_{-\pi}^{\pi} \phi(s) \sin ns\, ds,, \end{aligned} \tag{5.24}$$

for $n = 1, 2, ...$;
Therefore, the solution $u(r, \theta)$ of the boundary value problem for Laplace's equation in a disk is given by the formulae (5.23) and (5.24).
Example. Let us consider the following boundary value problem:

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad -\pi < \theta < \pi, \quad 0 < r < 2, \\ &u(2, \theta) = \cos\frac{\theta}{4}, \qquad\qquad -\pi \leq \theta \leq \pi, \end{aligned} \tag{5.25}$$

Solution. In order to find the solution $u(r, \theta)$, we apply the formulae (5.23) and (5.24). So that, we compute the Fourier coefficients of the function $\phi(\theta) = \cos\frac{\pi\theta}{4}$ given in the boundary condition. By formulae (5.24), we find

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{s}{4}\, ds = \frac{4\sqrt{2}}{\pi}, \\ a_n &= \frac{1}{2^n \pi}\int_{-\pi}^{\pi} \cos\frac{s}{4} \cos ns\, ds = -\frac{4(-1)^n\sqrt{2}}{2^n \pi (16n^2 - 1)}, \\ b_n &= \frac{1}{2^n \pi}\int_{-\pi}^{\pi} \cos\frac{s}{4} \sin ns\, ds = 0, \qquad\qquad n = 1, 2, ... \quad , \end{aligned} \tag{5.26}$$

Hence, the solution

$$u(r, \theta) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(16n^2 - 1)} r^n \cos n\theta,$$

5.7.1 Fundamental Solution of Laplace Equation

We shall find the fundamental solution of Laplace's equation. Here, we write the Laplace equations in the variables $X = (x_1, x_2)$, and $X = (x_1, x_2, x_3)$.

$$\begin{aligned} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} &= 0, \qquad n = 2 \\ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} &= 0, \quad n = 3, \end{aligned} \tag{5.27}$$

Let $n = 2$ and let $Y = (y_1, y_2)$ be a fixed point on x_1, x_2-plane. Then the distance of the point $Y = (y_1, y_2)$ from a point $X = (x_1, x_2)$ is

$$r(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We note that

$$\frac{\partial r}{\partial x_1} = \frac{x_1 - y_1}{r}, \qquad \frac{\partial r}{\partial x_2} = \frac{x_2 - y_2}{r}.$$

Let $r(X, Y) > 0$ and let $U(X, Y) = \ln \dfrac{1}{r(X, Y)}$. Now, we compute

$$\begin{aligned} \frac{\partial \ln \frac{1}{r}}{\partial x_1} &= -\frac{x_1 - y_1}{r^2}, & \frac{\partial \ln \frac{1}{r}}{\partial x_2} &= -\frac{x_2 - y_2}{r^2}, \\ \frac{\partial^2 \ln \frac{1}{r}}{\partial x_{11}^2} &= \frac{1}{r^2} - \frac{2(x_1 - y_1)^2}{r^4}, & \frac{\partial^2 \ln \frac{1}{r}}{\partial x_2^2} &= \frac{1}{r^2} - \frac{2(x_2 - y_2)^2}{r^4}. \end{aligned}$$

Hence, we find

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = 0.$$

Therefore, $U(X, Y) = \ln \dfrac{1}{r(X, Y))}$, $r(X, Y) > 0$ is the harmonic function on the whole x, y plane except the focus point $Y = (y_1, y_2)$. This function is called fundamental solution of the two dimensional Laplace equation.

.

Now, let $n = 3$ and $Y = (y_1, y_2, y_3)$ be a fixed point on the R^3 space. Then the distance of the point Y from the point $X = (x_1, x_2, x_3)$ is

$$r(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

We note that

$$\frac{\partial r}{\partial x_1} = \frac{x_1 - y_1}{r}, \qquad \frac{\partial r}{\partial x_2} = \frac{x_2 - y_2}{r}, \qquad \frac{\partial r}{\partial x_3} = \frac{x_3 - y_3}{r}.$$

Let $r(X,Y)) > 0,\ X = (x_1, x_2, x_3),\ Y = (y_1, y_2, y_3)$ and let $U(X,Y) = \dfrac{1}{r(X,Y)}$.
Then, we compute

$$\frac{\partial U}{\partial x_1} = -\frac{x_1 - y_1}{r^3}, \qquad \frac{\partial U}{\partial x_2} = -\frac{x_2 - y_2}{r^2}, \qquad \frac{\partial U}{\partial x_3} = -\frac{x_3 - y_3}{r^3}$$
$$\frac{\partial^2 U}{\partial x_2^2} = \frac{3(x_1 - y_1)^2}{r^5} - \frac{1}{r^3}, \quad \frac{\partial^2 U}{\partial x_2^2} = \frac{3(x_2 - y_2)^2}{r^5} - \frac{1}{r^3}, \quad \frac{\partial^2 u}{\partial x_3^2} = \frac{3(x_3 - y_3)^2}{r^5} - \frac{1}{r^3}$$

Hence, we find

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = 0.$$

Therefore, $U(X,Y) = \dfrac{1}{r(X,Y))},\ r(X,Y)) > 0$ is the harmonic function on the variables x_1, x_2, x_3 in the whole space except the focus point $Y = (y_1, y_2, y_3)$. This function is called fundamental solution of the three dimensional Laplace equation.

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Green's identities. Let Ω be a bounded domain in R^2 with the piecewise smooth closed boundary $\partial\Omega$. Let $u, v \in C^2(\Omega)$ be twice continuously differentiable functions in Ω. Then, the following second Green's identity holds:

$$\begin{aligned} n = 2, \quad \int\int_\Omega (v\Delta u - u\Delta v) d\sigma &= \int_{\partial\Omega} (u\frac{dv}{dn} - v\frac{du}{dn}) ds. \\ n = 3, \quad \int\int\int_\Omega (v\Delta u - u\Delta v) d\sigma &= \int\int_{\partial\Omega} (u\frac{dv}{dn} - v\frac{du}{dn}) ds. \end{aligned} \tag{5.28}$$

Here $\dfrac{du}{dn}$ denotes normal inner derivative to the boundary $\partial\Omega$. Note that for $n = 2$, $\partial\Omega$ is a curve on x_1, x_2 plane, and for $n = 3$, $\partial\Omega$ is a surface in the space R^3. Note that the both functions u and v are not assumed to be harmonic.

.

Proof. The prove of both Green's identities are similar. So, let us prove the Green's identity for $n = 2$. Integrating by parts, we have

$$\int\int_\Omega v\frac{\partial^2 u}{\partial x_1^2} dx_1 = -\int_{\partial\Omega} [v\frac{\partial u}{\partial x_1} - u\frac{\partial v}{\partial x_1}] \cos(n, x_1) ds + \int\int_\Omega u\frac{\partial^2 v}{\partial x_1^2} dx_1,$$
$$\int\int_\Omega v\frac{\partial^2 u}{\partial x_2^2} dx_2 = -\int_{\partial\Omega} [v\frac{\partial u}{\partial x_2} - u\frac{\partial v}{\partial x_2}] \cos(n, x_2) ds + \int\int_\Omega u\frac{\partial^2 v}{\partial x_2^2} dx_2.$$

Hence, by adding both sides and moving the term with Laplacian from right to left side, we obtain the Green's identity

$$\int_\Omega \int [v\Delta u - u\Delta v] d\sigma = \int_{\partial\Omega} [u\frac{dv}{dn} - v\frac{du}{dn}] ds,$$

where the normal derivative

$$\frac{du}{dn} = \frac{\partial u}{\partial x_1}\cos(n, x_1) + \frac{\partial u}{\partial x_2}\cos(n, x_2),$$

Here $\cos(n, x_1)$ $\cos(n, x_2)$ are directive cosines between normal vector n and x_1 and x_2 axes, respectively.

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corollary 1. For harmonic functions u and v in Ω, from Green;s identities, we obtain the following formulae:

$$\begin{aligned} &\int_{\partial\Omega}[v\frac{du}{dn} - u\frac{dv}{dn}]ds = 0, \qquad n = 2 \\ &\int\int_{\partial\Omega}[v\frac{du}{dn} - u\frac{dv}{dn}]ds = 0, \quad n = 3. \end{aligned} \tag{5.29}$$

In particular, when $v = 1$, we obtain next corollary
corollary 2. Every harmonic function u satisfies the following identity:

$$\begin{aligned} &\int_{\partial\Omega}\frac{du}{dn}ds = 0. \qquad n = 2, \\ &\int\int_{\partial\Omega}\frac{du}{dn}ds = 0, \quad n = 3. \end{aligned} \tag{5.30}$$

5.7.2 Theorem on representation of harmonic functions

Representation Theorem. Every harmonic function $u(X)$ in the bounded domain Ω with a smooth boundary $\partial\Omega$ satisfies the following formula

$$\begin{aligned} u(X) &= \frac{1}{2\pi}\int_{\partial\Omega}[u(Y)\frac{dU(X,Y)}{dn} - U(X,Y)\frac{du(Y)}{dn}]ds_Y, \qquad X \in \Omega, \quad n = 2 \\ u(X) &= \frac{1}{4\pi}\int\int_{\partial\Omega}[u(Y)\frac{dU(X,Y)}{dn} - U(X,Y)\frac{du(Y)}{dn}]ds_Y, \quad X \in \Omega, \quad n = 3. \end{aligned} \tag{5.31}$$

Proof. Let u be a harmonic function and $v = U$ be the fundamental solution of Laplace equation. Then, we cannot apply the formula (5.29) to these functions, since the fundamental solution $U(X, Y)$ has the singular point $X = Y \in \Omega$. However, we can apply the formula to the domain $\Omega_0 = \Omega - K$, where K is a disk when $n = 2$ or a ball when $n = 3$. So that, in Ω_0, we have

$$\begin{aligned} &\int_{\partial\Omega_0}[u\frac{dU}{dn} - U\frac{du}{dn}]ds = 0, \qquad n = 2 \\ &\int\int_{\partial\Omega_0}[u\frac{dU}{dn} - U\frac{du}{dn}]ds = 0, \quad n = 3. \end{aligned} \tag{5.32}$$

Now, let us write the above integrals along the boundary $\partial\Omega$ of Ω and the boundary ∂K of K. Taking into consideration negative orientation of ∂K with respect Ω, from (5.32), we have

$$\begin{aligned}
&\int_{\partial\Omega}[u\frac{dU}{dn} - U\frac{du}{dn}]ds = \int_{\partial K}[u\frac{dU}{dn} - U\frac{du}{dn}]ds, \qquad n = 2\\
&\int\int_{\partial\Omega}[u\frac{dU}{dn} - U\frac{du}{dn}]ds = \int\int_{\partial K}[u\frac{dU}{dn} - U\frac{du}{dn}]ds \quad n = 3.
\end{aligned} \tag{5.33}$$

Next, we compute the normal derivative of the fundamental solution $U(X,Y)$ to the boundary ∂K

$$U(X,Y) = \begin{cases} \ln\frac{1}{r}, & n = 2, \quad X = (x_1,x_2) \in \Omega_0, \qquad Y = (y_1,y_2) \in \partial K,\\ \frac{1}{r}, & n = 3, \quad X = (x_1,x_2,x_3) \in \Omega_0, \quad Y = (y_1,y_2,y_3) \in \partial K \end{cases}$$

Then, we compute

$$\frac{dU(X,Y)}{dn} = \begin{cases} \frac{1}{r}, & n = 2, \quad X = (x_1,x_2) \in \Omega_0, \qquad Y = (y_1,y_2) \in \partial K,\\ \frac{1}{r^2}, & n = 3, \quad X = (x_1,x_2,x_3) \in \Omega_0, \quad Y = (y_1,y_2,y_3) \in \partial K \end{cases}$$

Substituting the above formulae for the normal derivatives into (5.33), we obtain

$$\begin{aligned}
&\int_{\partial\Omega}[u\frac{dU}{dn} - U\frac{du}{dn}]\, ds = \frac{1}{r}\int_{\partial K} u\, ds - \ln\frac{1}{r}\int_{\partial K}\frac{du}{dn}ds, \qquad n = 2\\
&\int\int_{\partial\Omega}[u\frac{dU}{dn} - U\frac{du}{dn}]ds = \frac{1}{r^2}\int_{\partial K} u\, ds - \frac{1}{r}\int\int_{\partial K}\frac{du}{dn}ds \quad n = 3.
\end{aligned} \tag{5.34}$$

Hence, by corollary 2, we have

$$\begin{aligned}
&\int_{\partial\Omega}[U\frac{\partial u}{dn} - u\frac{dU}{dn}]ds = \frac{1}{r}\int_{\partial K} u(Y)ds_Y, \qquad n = 2\\
&\int\int_{\partial\Omega}[U\frac{du}{dn} - u\frac{dU}{dn}]ds = \frac{1}{r^2}\int_{\partial K} u(Y)ds_Y \quad n = 3.
\end{aligned} \tag{5.35}$$

Now, we apply the identity

$$\int_{\partial K} u(Y)ds_Y = \int_{\partial K}[u(Y) - u(X)]ds_Y + u(X)\int_{\partial K} ds_Y, \tag{5.36}$$

Because of uniform continuity of the harmonic function $u(X)$, we have

$$|u(Y) - u(X)| < \epsilon$$

for sufficiently small $r(X,Y)$.
Then, we have

$$\begin{aligned} &|\int_{\partial K}[u(Y)-u(X)]ds_Y| < 2\pi\epsilon\, r \qquad \text{for} \quad n=2 \\ &|\int\int_{\partial K}[u(Y)-u(X)]ds_Y| < 4\pi\epsilon\, r^2 \quad \text{for} \quad n=3. \end{aligned} \tag{5.37}$$

Using (5.36) and (5.37), we compute the limits

$$\begin{aligned} &\lim_{r->0}\frac{1}{r(X,Y)}\int_{\partial K}u(Y)ds_Y = 2\pi u(X) \qquad \text{for} \quad n=2, \\ &\lim_{r->0}\frac{1}{r^2(X,Y)}\int\int_{\partial K}u(Y)ds_Y = 4\pi u(X), \quad \text{for} \quad n=3. \end{aligned}$$

Hence by formula (5.35), we obtain equality (5.31). This ends the proof.
From the representative theorem, we conclude the following important formula concerning boundary value problems for harmonic functions. Namely, let us substitute to (5.31), $u(X) \equiv 1$. Then, we obtain the formula

$$\begin{aligned} \int_{\partial\Omega}\frac{dU(X,Y)}{dn}ds_Y &= 2\pi, \quad X\in\Omega, \quad n=2 \\ \int\int_{\partial\Omega}\frac{dU(X.Y)}{dn}ds_Y &= 4\pi, \quad X\in\Omega, \quad n=3. \end{aligned} \tag{5.38}$$

Gauss Mean Value Formula Let $\Omega = K$ be the disk ($n=2$) or the ball ($n=3$) with the radius R and the center at X. Then, on the surface of K, we have

$$\begin{aligned} &U(X,Y) = \ln\frac{1}{R}, \quad \frac{dU}{dn} = \frac{1}{R}, \quad n=2, \\ &U(X,Y) = \frac{1}{R}, \qquad \frac{dU}{dn} = \frac{1}{R^2}, \quad n=3 \end{aligned} \tag{5.39}$$

Hence, by the representative theorem, we obtain Gauss Mean Value Formula for harmonic functions

$$\begin{aligned} u(X) &= \frac{1}{2\pi R}\int_{\partial K}u(Y)ds_Y, \qquad X=(x_1,x_2)\in K, \qquad n=2, \\ u(X) &= \frac{1}{4\pi R^2}\int\int_{\partial K}u(Y)ds_Y, \quad X=(x_1,x_2,x_3)\in K, \quad n=3. \end{aligned} \tag{5.40}$$

5.7.3 Green's Function

. We note that every harmonic function $H(X,Y)$ of the variable Y, and at fixed X, satisfies the identity (see formula (5.29))

$$\begin{aligned} &\int_{\partial\Omega}[u(Y)\frac{dH(X,Y)}{dn} - H(X,Y)\frac{du(Y)}{dn}]ds = 0, \qquad n = 2 \\ &\int\int_{\partial\Omega}[u(Y)\frac{dH(X,Y)}{dn} - H(X,Y)\frac{du(Y)}{dn}]ds = 0, \quad n = 3. \end{aligned} \tag{5.41}$$

Let us choose the harmonic function $H(X,Y) = U(X,Y)$ for $Y \in \partial\Omega$ at fixed $X \in \Omega$. Then, the Green's function is

$$G(X,Y) = U(X,Y) - H(X,Y)$$

By the representative theorem (see formula (5.31)), Green's function satisfies identity

$$\begin{aligned} u(X) &= \frac{1}{2\pi}\int_{\partial\Omega}[u(Y)\frac{dG(X,Y)}{dn} - G(X,Y)\frac{du(Y)}{dn}]ds_Y, \qquad X \in \Omega, \quad n = 2 \\ u(X) &= \frac{1}{4\pi}\int\int_{\partial\Omega}[u(Y)\frac{dG(X,Y)}{dn} - G(X,Y)\frac{du(Y)}{dn}]ds_Y, \quad X \in \Omega, \quad n = 3. \end{aligned} \tag{5.42}$$

Because $G(X,Y) = 0$, for $Y \in \partial\Omega$, $X \in \Omega$, therefore, every harmonic function $u(X)$ in Ω, satisfies the identity

$$\begin{aligned} u(X) &= \frac{1}{2\pi}\int_{\partial\Omega}u(Y)\frac{dG(X,Y)}{dn}ds_Y, \qquad X \in \Omega, \quad n = 2 \\ u(X) &= \frac{1}{4\pi}\int\int_{\partial\Omega}u(Y)\frac{dG(X,Y)}{dn}ds_Y, \quad X \in \Omega, \quad n = 3. \end{aligned} \tag{5.43}$$

Hence, by the formula (5.43), we arrive at the following theorem:

Theorem 5.5 *If $G(X,Y)$ is the Green's function for Laplace's equation, then the solution $u(X)$ of the Dirichlet's problem*

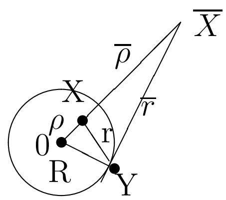
$$\Delta u(X) = 0, \quad X \in \Omega, \qquad u(X) = \phi(X), \quad X \in \partial\Omega$$

is given by the formula

$$\begin{aligned} u(X) &= \frac{1}{2\pi}\int_{\partial\Omega}\phi(Y)\frac{dG(X,Y)}{dn}ds_Y, \qquad X \in \Omega, \quad n = 2 \\ u(X) &= \frac{1}{4\pi}\int\int_{\partial\Omega}\phi(Y)\frac{dG(X,Y)}{dn}ds_Y, \quad X \in \Omega, \quad n = 3. \end{aligned} \tag{5.44}$$

Green's function for a disk with Dirichlet's condition. Let $r(0,Y) < R$, be the disk with the radius R and the center at the origin. Let us denote by

$$\rho = \overline{0X}, \quad \overline{\rho} = \overline{0\overline{X}}, \quad r = \overline{YX}, \quad \overline{r} = \overline{Y\overline{X}}.$$



Disk K

One can check that the Green's function for the disk K is

$$G(X,Y) = \begin{cases} \ln\frac{1}{r} - \ln(\frac{R}{\rho}.\frac{1}{\overline{r}}) = \ln\frac{\overline{r}\rho}{Rr}, & X \neq 0, \\ \ln\frac{1}{r} - \ln\frac{1}{R}, & X = 0 \end{cases} \tag{5.45}$$

Green's function for a ball with Dirichlet's conditions. Similarly, we construct Green's function for a ball $K(0,R) = \{X = (x_1, x_2, x_3) : \ r(0,X) < R\}$. Then, following the notations given above, we write the Green's function for the ball

$$G(X,Y) = \begin{cases} \frac{1}{r} - \frac{R}{\rho}.\frac{1}{\overline{r}}, & X \neq 0, \\ \frac{1}{r} - \frac{1}{R}, & X = 0 \end{cases} \tag{5.46}$$

Poisson's Integral. Let us note that in the case when the domain Ω is a disk K or a ball K, the Green's function is given by formulae (5.45) and (5.46). Then, the solution $u(X)$ of the Dirichlet's boundary value problem, in the case , is given by the Poisson's integral (see Representative Theorem, formulae (5.44)).

$$\begin{aligned} u(X) &= \frac{1}{2\pi}\int_{\partial K}\phi(Y)\frac{dG(X,Y)}{dn}ds_Y, \qquad X \in K, \quad n = 2 \\ u(X) &= \frac{1}{4\pi}\int\int_{\partial K}\phi(Y)\frac{dG(X,Y)}{dn}ds_Y, \quad X \in K, \quad n = 3. \end{aligned} \tag{5.47}$$

In order to express the solution $u(X)$ in a transparent form, we shall evaluate the kernel $\frac{\partial G(X,Y)}{\partial n_Y}$. Let $n = 3$ and $X \neq 0$. Then, we find

$$\frac{dG(X,Y)}{dn_Y} = -\frac{1}{r^2}\frac{dr}{dn_Y} + \frac{R}{\rho}.\frac{1}{\overline{r}^2}.\frac{d\overline{r}}{dn_Y} \tag{5.48}$$

Because

$$r = \sqrt{\sum_{i=1}^{3}(y_i - x_i)^2}$$

therefore

$$\frac{dr}{dn_Y} = \sum_{i=1}^{3}(y_i - x_i)^2\cos(n, x_i).$$

But $\cos(n, x_i) = -\frac{y_i}{R}$, so that

$$\frac{dr}{dn_Y} = \frac{1}{rR}(\sum_{i=1}^{3} x_i y_i - \sum_{i=1}^{3} y_i^2). \tag{5.49}$$

We note that $R^2 = \sum_{i=1}^{3} y_i^3$, and the expression $\sum_{i=1}^{3} x_i y_i$ is the inner product of the vectors $\overline{0X}$ and $\overline{0Y}$. Hence, it implies the identity

$$\sum_{i=1}^{3} x_i y_i = R\rho,$$

where ω is the angle between the vectors $\overline{0X}$ and $\overline{0Y}$. Then, we can write formula (5.49) in the form

$$\frac{dr}{dn_Y} = \frac{1}{r}(\rho\cos\omega - R). \tag{5.50}$$

In a similar way, we obtain the derivative $\frac{d\overline{r}}{dn_Y}$.
Taking into consideration the proportion

$$\frac{R}{\rho} = \frac{\overline{r}}{r} = \frac{\overline{\rho}}{R},$$

we obtain

$$\frac{d\overline{r}}{dn_Y} = \frac{1}{\overline{r}}(\overline{\rho}\cos\omega - R) = \frac{\rho}{Rr}(\frac{R^2}{\rho}\cos\omega - R) = \frac{1}{r}(R\cos\omega - \rho). \tag{5.51}$$

From the formulae (5.48), (5.50) and (5.51), we find

$$\begin{aligned} \frac{dG(X,Y)}{dn_Y} &= -\frac{1}{r^2}(\rho\cos\omega - R) + \frac{R}{r\overline{r}\rho}(R\cos\omega - \rho) \\ &= \frac{1}{r^3}[-(\rho\cos\omega - R) + \frac{\rho}{R}(R\cos\omega - \rho)] \end{aligned} \tag{5.52}$$

Hence, w obtain the equality

$$\frac{dG(X,Y)}{dn_Y} = \frac{R^2 - \rho^2}{Rr^3}, \qquad X \in K, \qquad m = 3. \tag{5.53}$$

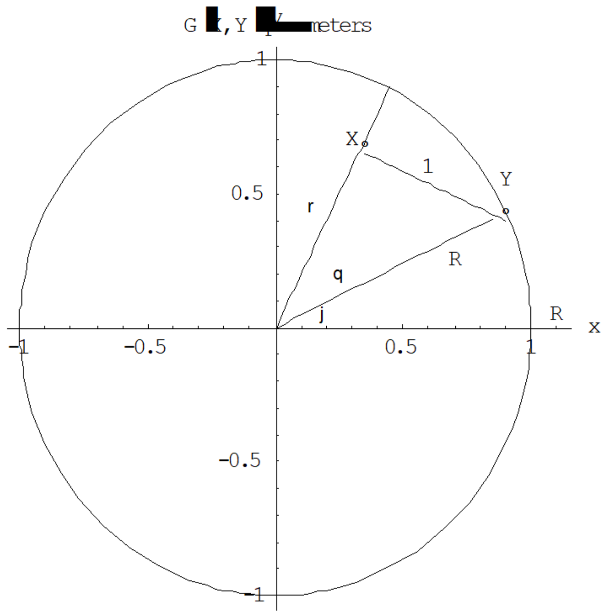
Therefore, Poisson's integral takes the following form:

$$u(X) = \frac{1}{2\pi R} \int_{\partial K} \frac{R^2 - \rho^2}{r^2} \phi(Y) ds_Y, \qquad X \in K, \quad n = 2. \tag{5.54}$$

and

$$u(X) = \frac{1}{4\pi R} \int\int_{\partial K} \frac{R^2 - \rho^2}{r^3} \phi(Y) ds_Y, \qquad X \in K, \qquad n = 3, \tag{5.55}$$

On the figure, we present the parameters r, R, ρ, ϕ and θ of the Green's function for the disk K.



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Let us write the Poisson's integral in the polar coordinates

$$y_1 = R\cos\varphi, \qquad y_2 = R\sin\varphi$$

. From the figure and by cosine formula, we find

$$r^2 = R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi).$$

Therefore, the kernel of the Poisson's integral

$$\frac{R^2 - \rho^2}{r^2} = \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)}. \tag{5.56}$$

When point Y is moving along the circle K with the radius R, at fixed point$X$, the angle $\varphi \in [0, 2\pi]$. Changing the variable of integration $Y = (y_1, y_2) \in \partial K$, to polar coordinates $y_1 = R\cos\varphi,\ y_2 = R\sin\varphi$ by (5.54) and (5.56), we obtain the solution of the boundary value problem given by the Poisson's integral

$$u(\rho,\theta) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\varphi)}\phi(\varphi)d\varphi. \tag{5.57}$$

We can interpret the Poisson integral solution (5.57) as finding the potential u at (ρ,θ) as a weighted average of the boundary potential $\phi(\theta)$ weighted by the Poisson's kernel (5.56). This tells us something about physical systems: namely that the potential at a point is the weighted average of neighboring potentials. The Poisson's kernel tells just how much weight to assign to each point.

Let us note that the potential at the center of the circle is given by the formula

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\varphi)\, d\varphi. \tag{5.58}$$

Example 1. Evaluate the potential at the center of the circle by Poisson integral for the boundary given function $\phi(\varphi) = \cos\dfrac{\varphi}{4},\ 0 \le \varphi \le 2\pi$.

Solution. By formula (5.57), we compute

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\varphi)d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \cos\frac{\varphi}{4}\, d\varphi = \frac{2}{\pi}.$$

Example 2. Show that the integral of Poisson kernel is equal to 2π, that is

$$\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\varphi)}d\varphi = 2\pi. \tag{5.59}$$

Solution. Let the boundary given function $\phi(\varphi) \equiv 1$. Then, the solution of the boundary problem $u(X) = 1$ for all $X \in K$. Thus, by formula (5.57), we get

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(\theta-\varphi)}\, d\varphi.$$

Hence

$$\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(\theta-\varphi)}\, d\varphi = 2\pi.$$

Example 3. Solve the following boundary value problem using Poisson integral

$$u_{xx} + u_{yy} = 0, \quad (x,y) \in K = \{x^2+y^2 < 4\},$$

$$u(x,y) = 2, \qquad (x,y) \in \partial K = \{x^2+y^2 = 4\},$$

Solution . By the Poisson formula, the solution in the polar coordinates is

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{4-\rho^2}{4+\rho^2-2*4\cos(\theta-\varphi)}2\, d\varphi,$$

Hence, by formula (5.59), we compute

$$u(\rho,\theta) = \frac{2}{2\pi}\int_0^{2\pi} \frac{4-\rho^2}{4+\rho^2-2*4\cos(\theta-\varphi)}\,d\varphi = 2.$$

In the Cartesian coordinates, we have also the constant solution $u(x, y) = 2$ for all $(x, y) \in K$.

5.8 Helmholz Equation and Eigenvalue Problem.

We shall solve the following Helmholz equation in the polar coordinates (r, θ), $x = r\cos\theta$, $y = r\sin\theta$ with homogeneous boundary conditions:

$$\begin{aligned} &\Delta u(r,\theta) + \lambda^2 u(r,\theta) = 0, \\ &u(1,\theta) = 0, \qquad 0 \le \theta 2\pi, \end{aligned} \tag{5.60}$$

where

$$\Delta u(r,\theta) \equiv u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Applying the method of separation of variables let us substitute to Helmholz equation

$$u(r,\theta) = R(r)\Theta(\theta).$$

Then, we obtain

$$\begin{aligned} &r^2R'' + rR' + (\lambda^2r^2 - n^2))R = 0, \qquad \textit{Bessel's equation} \\ &R(1) = 0, \\ &\Theta'' + n^2\Theta = 0. \end{aligned} \tag{5.61}$$

Bessel's Equation Now, we shall solve the ordinary differential Bessel's equation

$$\begin{aligned} &r^2R'' + rR' + (\lambda^2r^2 - n^2))R = 0, \qquad 0 < r < 1, \\ &R(1) < \infty, \qquad \textit{physical condition}, \\ &R(1) = 0, \\ &\Theta'' + n^2\Theta = 0. \end{aligned} \tag{5.62}$$

As we know from the theory of ordinary differential equations, that Bessel's equation has two linearly independent solutions

1. $R_1(r) = AJ_n(\lambda r)$, n-th order Bessel function of the first kind,
2. $R_2(r) = BY_n(\lambda r)$, n-th order Bessel function of the second kind.

Here, there are power series representation of the Bessel functions

$$J_n(\lambda r) = (\frac{\lambda r}{2})^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}(\frac{\lambda r}{2})^{2m},$$
$$Y_n(\lambda r) = \frac{2}{\pi}\ln\frac{\lambda r}{2} J_n(\lambda r) - \frac{1}{\pi}(\frac{\lambda r}{2})^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}[\frac{(\lambda r)^2}{4}]^k.$$

Since $Y_n(\lambda r)$ is unbounded at $r = 0$, we choose as our solution

$$R(r) = AJ_n(\lambda r).$$

Next, we find $R(r)$ and λ using the boundary condition $R(1) = 1$. Namely, substituting $R(1) = 0$ into $AJ_n(\lambda r)$, we obtain

$$J_n(\lambda) = 0.$$

In other words, in order to be $R(r) = 0$ on the boundary ∂K of the circle K, we must pick the separation constant λ to be one of roots of the equation $J_n(r) = 0$, that is

$$\lambda = k_{nm},$$

where k_{nm} is the m-th root of $J_n(r) = 0$. Finally, we obtain the solution

$$u_{nm}(r, \theta) = AJ_n(k_{nm}r).$$

5.9 Exercises

Question 1. Consider the following boundary value problem:

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega = \{(x, y) : \ 0 < x < 1, \ 0 < y < 1\}, \tag{5.63}$$

Find the range of the values of the solution $u(x, y)$. which satisfies one of the boundary condition

(a) $u(x, y) = sin\pi xy, \qquad (x, y) \in \partial\Omega,$

(b) $u(x, y) = \cos \pi x\, y, \qquad (x, y) \in \partial\Omega,$

(c) $u(x, y) = \sin \pi x + \sin \pi y, \qquad (x, y) \in \partial\Omega,$

(d) $u(x, y) = \cos \pi x + \cos \pi y, \qquad (x, y) \in \partial\Omega,$

Question 2. Show that the following boundary value problem is stable

$$\begin{aligned} &u_{xx} + u_{yy} - u = f(x, y), \quad (x, y) \in \Omega = \{(x, y) : \ -1 < x < 1, \ -1 < y < 1\}, \\ &u(x, y) = \phi(x, y), \qquad ((x, y) \in \partial\Omega. \end{aligned} \tag{5.64}$$

for given continuous functions $f(x,y)$ and $\phi(x,y)$. Give an priori estimate of the solution $u(x,y)$, when $f(x,y) = e^{-x^2-y^2}$, $(x,y) \in \Omega$ and $\phi(x,y) = 0$, $(x,y) \in \partial\Omega$.

Question 3. Solve the following boundary value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && (x,y) \in \Omega = \{(x,y) : \ 0 < x < 1, \quad 0 < y < 2\}, \\ &u(x,0) = 0, && 0 \le x \le 1 \\ &u(0,y) = u(1,y) = 0, && 0 \le y \le 2, \\ &u(x,2) = \sin \pi x, && 0 \le x \le 1. \end{aligned} \tag{5.65}$$

Question 4. Solve the following boundary value problem:

$$\begin{aligned} &u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, && -\pi < \theta < \pi, \quad 0 < r < 4, \\ &u(4,\theta) = \cos \frac{\theta}{2}, && -\pi \le \theta \le \pi, \end{aligned} \tag{5.66}$$

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